Helmholtz equation - unbounded space

\[(\nabla^2 + k^2) G_k(\vec{r}, \vec{r}') = 0 \]

The solution to this equation is an outgoing spherical wave

\[ G_k(\vec{r}, \vec{r}') = G_k(\vec{r} - \vec{r}') = -\frac{1}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \]

This may be directly verified: Consider a small sphere, of radius \( \varepsilon \), centered on \( \vec{r}' \):

\[
S: \quad \vec{r} = \vec{r}' - \vec{r}'
\]

\[
\int (d\vec{r})(\nabla^2 + k^2) G \approx \int (d\vec{r}) \nabla^2 \left[ -\frac{1}{4\pi} \frac{e^{ik|\vec{r}|}}{|\vec{r}|} \right]
\]

\[
\approx \int d\Omega \rho^2 \frac{\partial}{\partial \rho} \left[ -\frac{1}{4\pi} \frac{e^{ik\rho}}{\rho} \right] \bigg|_{\rho = \varepsilon}
\]

\[
\approx \frac{1}{4\pi} \int d\Omega \rho^2 \frac{1}{\rho^2} e^{ik\rho} \bigg|_{\rho = \varepsilon} = 1
\]

And for \( r \neq r' \) \((\nabla^2 + k^2) G_k = 0\) is easily seen.
Alternatively, we may construct $G$ from the eigenfunction expansion:

\[ G_k(\vec{r}-\vec{r}') = \sum_n \frac{\psi_n^*(\vec{r}') \psi_n(\vec{r})}{\lambda_n - \lambda} \]

where $-\lambda = k^2$, $-\lambda_n = k'^2$, while

\[ (\nabla^2 + k'^2) \frac{\psi_k}{k'}(\vec{r}) = 0 \]

or \[ \frac{\psi_k}{k'}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}' \cdot \vec{r}} \]

(The $(2\pi)^{-3/2}$ factor is for normalization. The spectrum of eigenvalues is continuous, so

\[ \sum_{\lambda} \rightarrow \int (d\vec{k}') \]

and

\[ G_k(\vec{r}-\vec{r}') = \int (d\vec{k}') \frac{e^{-i\vec{k}' \cdot \vec{r}'}}{\frac{i}{k^2 - k'^2}} e^{i\vec{k} \cdot \vec{r}} \]

Do the integral in spherical coordinate

\[ (d\vec{k}') = k'^2 dk'd\phi'd\mu', \mu' = \cos \theta' \]

$z$-axis along direction of $\vec{r}-\vec{r}'$.
\[ G(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3} \int_0^\infty dk' k'^2 \int_0^{2\pi} d\phi' \int_0^1 dv' \frac{e^{ik'\rho}}{k^2 - k'^2} \]

where we replaced \( \int_0^\infty \) by \( \frac{1}{2} \int_{-\infty}^{\infty} \) since integrand is even in \( k' \). We evaluate this integral by contour methods.

Since now \( k \) coin sides with an eigenvalue we must choose the contour to define the Green's function. Suppose we choose the contour as shown, phasing below the pole at \( k \), above the pole \( k' \) at \( -k \). We close the contour in the UHP for the \( e^{ik\rho} \) term, in the LHP for the \( e^{-ik'\rho} \) term.
Then, by Jordan's lemma,

\[ G_{z} (\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^2} \frac{1}{2} \left\{ -\frac{2\pi i}{2k} \frac{ke^{ikr}}{i\rho} \right. \]

\[ + \frac{2\pi i}{-2k} \left( \frac{k}{i\rho} \right) e^{ikr} \rho^2 \]

\[ = -\frac{1}{4\pi} \frac{e^{ikr}}{\rho} \cdot Q.E.D. \]

If a different contour had been chosen, we would have got a different Green's function, not one corresponding to outgoing spherical waves. Boundary conditions uniquely determine the contour.

Note: \( G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r}) \), even though \( G_{z} \) is complex. The self-adjointness property from eigenfunction expansion is only formal, and is 'spoiled' by contour choice.
Green's Function for Scalar Wave Equation

The inhomogeneous scalar wave equation

\[
(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \psi(\vec{r}, t) = \rho(\vec{r}, t)
\]

requires boundary and initial conditions. The boundary conditions may be Dirichlet, Neumann, or mixed. The initial conditions are Cauchy (see p. 137). Thus we must specify \( \psi(\vec{r}, t_0) \), \( \frac{\partial}{\partial t} \psi(\vec{r}, t_0) \) at every point in the region being considered.

The Green's function \( G(\vec{r}, t; \vec{r}', t') \) satisfies

\[
(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) G(\vec{r}, t; \vec{r}', t') = \delta(\vec{r} - \vec{r}')
\]

\[
\times \delta(t - t')
\]

It must satisfy the homogeneous form of the boundary conditions satisfied by \( \psi \). Thus if \( \psi \) has a specified value everywhere on the bounding surface, the corresponding Green's function
must vanish on the surface. In classical physics, it is customary to adopt as initial conditions:

\[
\begin{align*}
G(\vec{r}, t, \vec{r}', t') \quad &\quad \big\{ \begin{array}{l}
G(\vec{r}, t, \vec{r}', t') \\
\frac{\partial G}{\partial t} (\vec{r}, t; \vec{r}', t')
\end{array} \big\} = 0 \quad \text{if } t < t' \\
\end{align*}
\]

These then define the so-called retarded Green's functions. They ensure that an effect occurs after its cause. [In fact, however, the time asymmetry of the Green's function, which is not present in the wave equations, is not necessary: and in fact it is impossible to maintain in relativistic quantum mechanics.]

With this Green's function what takes the place of the self-adjointness property given on p. 130? Since \( \frac{\partial G}{\partial t} \) is invariant when \( t \to -t \), we have

\[
\begin{align*}
a) \quad \nabla^2 G(\vec{r}; t; \vec{r}', t') - \frac{i}{c^2} \frac{\partial^2}{\partial t^2} G(\vec{r}; t; \vec{r}', t') &= \delta(\vec{r} - \vec{r}') \delta(t - t') \\
b) \quad \nabla^2 G(\vec{r}; t; \vec{r}'', t'') - \frac{i}{c^2} \frac{\partial^2}{\partial t^2} G(\vec{r}; t; \vec{r}'', t'') &= \delta(\vec{r} - \vec{r}'') \delta(t - t'')
\end{align*}
\]
Multiply a) by \( G(r_j - t; \vec{r}_\prime - t') \), b) by 
\( G(\vec{r}_j, \vec{r}_\prime, t') \), subtract, and integrate over 
the volume being considered, and over \( t \) from 
\(-\infty \) to \( T \), where \( T > t', t'' \), we have

\[
\int_{-\infty}^{T} dt \int d\vec{r} \left\{ G(\vec{r}_j, \vec{r}_\prime, t') \nabla^2 G(\vec{r}_j - t; \vec{r}_\prime - t') - G(\vec{r}_j - t; \vec{r}_\prime - t') \nabla^2 G(\vec{r}_j, \vec{r}_\prime, t') - G(\vec{r}_j, \vec{r}_\prime, t') \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(\vec{r}_j - t, \vec{r}_\prime - t') \right. \\
\left. + G(\vec{r}_j - t, \vec{r}_\prime - t') \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(\vec{r}_j, \vec{r}_\prime, t') \right\}
\]

\[=- G(\vec{r}_j^1 - t'; \vec{r}_\prime^1 - t'') + G(\vec{r}_j^1, t''; \vec{r}_\prime^1, t') \]

Now use Green's theorem, together with the identity

\[
\frac{\partial}{\partial t} \left( A \frac{\partial}{\partial t} B - B \frac{\partial}{\partial t} A \right) = A \frac{\partial^2}{\partial t^2} B - B \frac{\partial^2}{\partial t^2} A
\]

to conclude

\[=- G(\vec{r}_j^1 - t'; \vec{r}_\prime^1 - t'') + G(\vec{r}_j^1, t''; \vec{r}_\prime^1, t') \]

\[= \int_{-\infty}^{T} dt \int d\vec{r} \cdot \left\{ G(\vec{r}_j, \vec{r}_\prime, t') \nabla G(\vec{r}_j - t; \vec{r}_\prime - t') \right. \\
\left. - G(\vec{r}_j - t; \vec{r}_\prime - t') \nabla G(\vec{r}_j, \vec{r}_\prime, t') \right\} \]
\[-\int V \left\{ \frac{1}{c^2} \left[ \frac{\partial}{\partial t} G(\mathbf{r}, t; \mathbf{r}', t') \right. \right. \left. \frac{\partial}{\partial t} G(\mathbf{r}, -t; \mathbf{r}', -t') \right. \right. \left. \frac{\partial}{\partial t} G(\mathbf{r}, -t; \mathbf{r}', t') \right]\right\}\left. \right|_{t=0}^{t=T} \]

The surface integral vanishes, since both Green's functions satisfy the same homogeneous boundary conditions on \( S \). (B.C. are time independent.) The second integral is zero because

\[ G(\mathbf{r}, -\infty; \mathbf{r}', t') = 0 \]

\[ \frac{\partial G(\mathbf{r}, -\infty; \mathbf{r}', t')}{\partial t} \]

and

\[ G(\mathbf{r}, -T; \mathbf{r}', -t') = 0 \]

\[ \frac{\partial G(\mathbf{r}, -T; \mathbf{r}', -t')}{\partial t} \]

since \(-T < -t'\)

Thus, \( G(\mathbf{r}, t; \mathbf{r}', t') = G(\mathbf{r}, -t'; \mathbf{r}', -t) \)

the "reciprocity relation."

How do we express a solution to the wave equation in terms of the Green's function?
The procedure is the same as in the derivation on p. 148. The field, and the Green’s fn., satisfy

\[ \nabla^2 \Phi(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi(\vec{r}, t) = \rho(\vec{r}, t) \]

\[ \nabla^2 G(\vec{r}, \vec{r}'; t, t') - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(\vec{r}, \vec{r}'; t, t') \]

\[ \left( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) = \delta(\vec{r} - \vec{r}') \delta(t - t') \]

Note the differentiations on \( G \) are with respect to the second set of arguments (this equation follows from the reciprocity relation). Again multiply both sides by \( G(\vec{r}, \vec{r}'; t, t') \) and by \( \Phi(\vec{r}, t') \), subtract, integrate over the volume, and over \( t \), from \( t_0 \) to \( t^+ \) (\( t^+ \) means \( t + \epsilon \), \( \epsilon > 0 \)), \( \epsilon \to 0 \) \((t_0 < t) \) (\( \vec{r} \) inside \( V \))

\[ \int_{t_0}^{t_+} dt' \int (d\vec{r}') \{ G(\vec{r}, \vec{r}'; t, t') \nabla^2 \Phi(\vec{r}', t') \]

\[ - \Phi(\vec{r}, t') \nabla^2 G(\vec{r}, \vec{r}'; t, t') \]

\[ - \frac{1}{c^2} \left[ G(\vec{r}, \vec{r}'; t, t') \frac{\partial^2}{\partial t^2} \Phi(\vec{r}, t') \right] \]

\[ - \Phi(\vec{r}, t') \frac{\partial^2}{\partial t^2} G(\vec{r}, \vec{r}'; t, t') \} \]

\[ = - \Psi(\vec{r}, t) + \int_{t_0}^{t_+} dt' \int (d\vec{r}') G(\vec{r}, \vec{r}'; t, t') \]

\[ \times \Phi(\vec{r}', t') \]
Use Green's theorem, and identity on p. 16, to conclude

\[ \psi(r, t) = \int_{t_0}^{t_+} dt' \int (d\vec{r}') G(r, t; \vec{r}', t') \rho(\vec{r}', t') \]

\[ - \int_{t_0}^{t_+} dt' \oint_{S} \left\{ \nabla' \cdot G(r, t; \vec{r}', t') \nabla' \psi(\vec{r}', t') \right\} \]

\[ - \frac{1}{c^2} \int (d\vec{r}') \left[ G(r, t; \vec{r}', t_0) \frac{\partial}{\partial t_0} \psi(\vec{r}', t_0) \right] \]

\[ - \psi(\vec{r}', t_0) \frac{\partial}{\partial t_0} G(r, t; \vec{r}', t_0) \]

Interpretation:

1. 1st integral represents the effect of sources \( \rho \) distributed throughout the volume \( V \).

2. 2nd integral represents the B.C. \( \nabla \psi \) specifies homogeneous Neumann B.C. on \( S \), so

\[ \vec{n} \cdot \nabla \psi |_S = f(\vec{r}') \] is specified

then use homogeneous Neumann B.C. for \( G \)
\[ \mathbf{n} \cdot \nabla G(\mathbf{r}, t ; \mathbf{r}', t') = 0. \]

So 2nd integral reads

\[ - \int_{t_0}^{t+} dt' \int d\mathbf{r}' \cdot G(\mathbf{r}, t ; \mathbf{r}', t') \nabla' \Psi(\mathbf{r}', t') \]

\[ - \mathbf{n} \cdot \nabla' \Psi(\mathbf{r}', t') \text{ represents a surface source distribution} \]

[Other types of B.C. are as discussed on pp. 141-143]

3rd integral represents the effect of the initial conditions.

\( \Psi(\mathbf{r}', t_0) \), \( \frac{\partial}{\partial t_0} \Psi(\mathbf{r}', t_0) \)

are specified. They correspond to impulsive sources at \( t = t_0 \).

\[ P_{\text{init}}(\mathbf{r}, t') = - \frac{1}{\varepsilon^2} \left[ \frac{\partial}{\partial t_0} \Psi(\mathbf{r}', t_0) \delta(t'-t_0) \right. \]

\[ + \left. \Psi(\mathbf{r}', t_0) \delta'(t'-t_0) \right] \]

(verify this by integrating by parts, and letting lower limit of \( t' \) integral be \( t_0-\varepsilon \))
Wave equation in unbounded space

We solve

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\vec{r}, t; \vec{r}', t') = \delta(\vec{r} - \vec{r}') \times \delta(t - t') \]

by noting \( G \) is a function of \( \vec{r} - \vec{r}', t - t' \) only,

\[ G(\vec{r} - \vec{r}', t - t') \], and

introducing the Fourier transform

\[ g(\vec{k}, \omega) = \int (d\vec{r}) \, d\tau \, e^{i \vec{k} \cdot \vec{r}} \, e^{-i \omega \tau} \, G(\vec{r}, \tau) \]

The Fourier transform of the Green's function equation is \( (c = 1 \text{ for convenience}) \)

\[ \left[ -\kappa^2 + \omega^2 \right] g(\vec{k}, \omega) = 1 \]

or \[ g(\vec{k}, \omega) = \frac{1}{\omega^2 - \kappa^2} \] \( (\kappa^2 = \vec{k} \cdot \vec{k}) \)

Then

\[ G(\vec{r}, \tau) = \int (d\vec{k}) \, d\omega \, \frac{1}{2\pi} e^{-i\vec{k} \cdot \vec{r}} e^{i\omega \tau} g(\vec{k}, \omega) \]

\[ = \int (d\vec{k}) \, d\omega \, \frac{1}{2\pi} e^{-i\vec{k} \cdot \vec{r}} e^{i\omega \tau} \frac{1}{\omega^2 - \kappa^2} \]
the \( w \) integral is not well defined until we impose the boundary condition

\[ G(\bar{\rho}, \tau) = 0 \quad \text{if} \quad \tau < 0. \]

This will be true if the poles are located above the real \( \omega \) axis.

\[
\int_{-k+i\epsilon}^{k+i\epsilon} \frac{e^{i\omega \tau}}{(\omega-k)(\omega+k)} \, d\omega
\]

Close contour in UHP for \( \tau > 0 \); in LHP for \( \tau < 0 \); in both cases, by Jordan's lemma, the infinite semicircle gives no contribution.

\[
\int_{-\infty}^{\infty} \frac{e^{i\omega \tau}}{(\omega-k)(\omega+k)} \, d\omega
\]

\[
= \begin{cases} 
  i \left[ e^{ikt} \frac{1}{2k} - e^{-ikt} \frac{1}{2k} \right] & \tau > 0 \\
  0 & \tau < 0 
\end{cases}
\]

Thus, if \( \tau > 0 \),

\[
G(\bar{\rho}, \tau) = \frac{1}{(2\pi)^3} \int_{0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik\rho \mu} \\
\times \frac{i}{2\pi} \left[ e^{ikt} - e^{-ikt} \right]
\]
\[
\frac{1}{(2\pi)^2} \int_0^\infty \frac{k \, dk}{2i \chi k} \left[ e^{ik\rho} - e^{-ik\rho} \right] i \left[ e^{ikt} - e^{-ikt} \right] = \frac{1}{(2\pi)^2} \frac{1}{2 \rho} \frac{1}{2} \int_{-\infty}^\infty dk \left[ e^{i k(\rho + \tau)} + e^{-i k(\rho + \tau)} \right] \\
- e^{-i k(\rho - \tau)} - e^{-i k(\tau - \rho)}
\]
\[
= \frac{1}{(2\pi)^2} \frac{1}{4 \rho} 2\pi \left\{ 2 \delta(\rho + \tau) - 2 \delta(\rho - \tau) \right\}
\]

But \( \rho, \tau \) are both positive, so \( \rho + \tau \) can never vanish.

So
\[
G(\vec{r}, \tau) = -\frac{1}{4\pi} \frac{1}{\rho} \delta(\rho - \tau)
\]

or restoring \( \chi \),
\[
G(\vec{r} - \vec{r}', t - t') = -\frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|} \times
\]
\[
\chi \delta\left(\frac{|\vec{r} - \vec{r}'|}{c} - (t - t')\right)
\]

The effect at \( \vec{r} \) at time \( t \) is due to the effect of the point source at \( \vec{r}' \) at time \( t' = t - \frac{|\vec{r} - \vec{r}'|}{c} \).
Physically, this means that the "signal" propagates with speed $c$.

Let us make this more concrete by considering a simple example, a point "charge" moving with velocity $\vec{v}$ (unit).

Then $p(\vec{r}, t) = q \delta(\vec{r} - \vec{v} t)$

\[ \frac{\vec{v}}{c} \rightarrow \]

There are no effects from the infinite surface, so

\[ \gamma(\vec{r}, t) = \int_{t_0}^{t_+} \int (d\vec{r}') G(\vec{r} - \vec{r}', t-t') \rho(\vec{r}', t') \]

\[ = -\frac{q}{4\pi} \int_{t_0}^{t_+} dt' \frac{1}{|\vec{r} - \vec{v} t'|} \delta \left[ \frac{|\vec{r} - \vec{v} t'|}{c} - (t-t') \right] \]

Let $p = \frac{|\vec{r} - \vec{v} t'|}{c} + t'$

\[ dp = dt' \left[ \frac{v^2 t' - \vec{r}, \vec{v}}{c |\vec{r} - \vec{v} t'|} + 1 \right] \]

since $|\vec{r} - \vec{v} t'| = \sqrt{r^2 + v^2 t'^2 - 2 \vec{r}, \vec{v} t'}$
\[ \psi(r, t) = -\frac{q}{4\pi} \int \frac{d\rho s(\rho - t)}{v^2 t' - \vec{r} \cdot \vec{v} + |\vec{r} - \vec{v} t'|} \]

\[ = -\frac{q}{4\pi} \left. \frac{1}{\frac{1}{c}(v^2 t' - \vec{r} \cdot \vec{v}) + |\vec{r} - \vec{v} t'|} \right|_{\rho = t} \]

The equation \( \rho = t \) determines \( t' \), the "retarded time."

A more compact way of writing this is to let
\[ \vec{\rho} = \vec{r} - \vec{v} t' \], so

\[ \psi(\vec{r}, t) = -\frac{q}{4\pi} \frac{1}{\rho - \frac{\vec{v} \cdot \vec{\rho}}{c}} \]

\( \vec{\rho} \) is the displacement of observation point (or from source when signal now reaching observer was radiated) then, at time \( \rho/c \) earlier.
\[ h(r, t) = \frac{1}{4\pi} \int_{t_0 - \infty}^{t^*} dt' \frac{1}{\rho(t')} \delta \left( \frac{\rho(t')}{c} - (t - t') \right) \]

\[ t = \frac{\rho(t')}{c} + t' \]

\[ d\tau = dt' \left( 1 + \frac{d\rho}{dt'} \frac{1}{c} \right) \]

\[ \rho = \sqrt{r^2 + v^2 t'^2 - 2r \cdot v \cdot t'} \]

\[ \frac{d\rho}{dt'} = \frac{v^2 t' - r \cdot \overrightarrow{v}}{\rho(t')} = -\frac{\overrightarrow{v} \cdot \overrightarrow{r}}{\rho} \]

\[ (= \frac{1}{2\rho} \frac{d}{dt'} \overrightarrow{r} \cdot \overrightarrow{v} = \frac{\overrightarrow{r}}{\rho} \cdot \overrightarrow{v} ) \]

\[ d\tau = dt' \left( 1 + \frac{\overrightarrow{v} \cdot \overrightarrow{r}}{\rho c} \right) \]

\[ \therefore \psi(\overrightarrow{r}; t) = -\frac{q}{4\pi} \int_{t_0 - \infty}^{t^*} d\tau \frac{1}{\rho(t) \left( 1 + \frac{\overrightarrow{r} \cdot \overrightarrow{v}}{\rho c} \right)(t)} \delta (\tau - t) \]

\[ = -\frac{q}{4\pi} \frac{1}{\rho(t) - \frac{\overrightarrow{r} \cdot \overrightarrow{v}}{c}(t)} \]

\[ \tau = t \] determines "retarded time" \( t \)

\[ \rho = \text{distance from source to observation pt at time } t' \left( = t - \frac{\rho}{c} \right) \]