Types of Boundary Conditions

Three types of 2nd order, homogeneous differential equations are commonly encountered in physics: (the dimensionality of \( r \) is not important)

**Hyperbolic:** \( \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u (\vec{r}, t) = 0 \)  
(wave eqn)

**Elliptic:** \( \left( \nabla^2 + k^2 \right) u (\vec{r}) = 0 \)  
(Helmholtz eqn, including Laplace)

**Parabolic:** \( \left( \nabla^2 - \frac{1}{K} \frac{\partial}{\partial t} \right) T (\vec{r}, t) = 0 \)  
(diffusion eqn)

The types of boundary conditions specified on which types of boundaries are necessary to uniquely specify a solution to these equations is as follows:

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Laplace B.C.: specify $u$, $\frac{\partial u}{\partial n}$ both on boundary ($\frac{\partial u}{\partial n}$ mean normal derivative to surface)

Dirichlet: specify $u$ only on surface

Neumann: specify $\frac{\partial u}{\partial n}$ only on surface

Mixed: specify $\alpha u + \beta \frac{\partial u}{\partial n}$ on surface

If the specified boundary values are zero, the B.C. are called homogeneous; otherwise inhomogeneous.

Example: To determine the motion of a string, described by

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) u = 0.$$

Must specify $u(x,0)$, $\frac{\partial u}{\partial t}(x,0)$ at some initial time ($t=0$)

This is open surface in ($ct$, $x$) plane.
Typically, one determines the eigenfunctions of a differential operator subject to homogeneous boundary conditions. That means that the Green's function obeys the same boundary conditions [see p. 136]. But suppose we seek a solution of

$$(L - \lambda) \psi = S$$

subject to inhomogeneous boundary conditions.

It cannot then be true that

$$\psi(\vec{r}) = \int (d\vec{r}') G(\vec{r}, \vec{r}') S(\vec{r}')$$

To see how to deal with this situation, let us consider the example of the (3-dim. Helmholtz equation):

$$(a) \quad (\nabla^2 + k^2) \psi(\vec{r}) = S(\vec{r})$$

We seek a solution $\psi(\vec{r})$ subject to arbitrary inhomogeneous (Dirichlet, Neumann, or mixed) boundary conditions on a surface $\partial V$ enclosing the volume $V$ of interest. The Green's function $G$ for this problem satisfies

$$(b) \quad (\nabla^2 + k^2) G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}')$$

subject to homogeneous boundary condition of the same type as $\psi$ satisfies.

Multiply (a) by $G$, (b) by $\psi$, ...
subtract, and integrate over the appropriate variables:

\[
\int d\mathbf{f}' \left[ G(\mathbf{r}, \mathbf{r}') (\nabla^2 + k^2) \psi(\mathbf{r}') \\
- \psi(\mathbf{r}') (\nabla^2 + k^2) G(\mathbf{r}, \mathbf{r}') \right]
\]

\[
= \int d\mathbf{f}' \left[ G(\mathbf{r}, \mathbf{r}') \nabla' \psi(\mathbf{r}') \\
- \psi(\mathbf{r}') \nabla' G(\mathbf{r}, \mathbf{r}') \right]
\]

Here we have interchanged \( \mathbf{r} \) and \( \mathbf{r}' \) in (a) and (b), and have used

\[
G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})
\]

[we assume eigenfunctions and Green's functions are real]. Now we use Green's theorem (Assignment 9, prob 2) to establish

\[
-\int d\mathbf{f}' \cdot \left[ G(\mathbf{r}, \mathbf{r}') \nabla' \psi(\mathbf{r}') \\
- \psi(\mathbf{r}') \nabla' G(\mathbf{r}, \mathbf{r}') \right]
\]

\[
+ \int d\mathbf{f}' \ G(\mathbf{r}, \mathbf{r}') \nabla' G(\mathbf{r}, \mathbf{r}') = \int \psi(\mathbf{r}), \nabla' \psi \nabla' G(\mathbf{r}, \mathbf{r}') S(\mathbf{r}') = \begin{cases} \psi(\mathbf{r}), \nabla' \psi & \text{in } V \\ 0, \text{ outside } V \end{cases}
\]

where in the surface integral \( d\mathbf{f}' \) is the outwardly directed surface element, and \( \mathbf{r}' \) lies within the surface \( \partial S \). This generalizes
the simple relation between "field" and "source" given on p. 135

How do we use this result? We always suppose \( \psi \) satisfies homogeneous B.C. on \( \partial V \).
If \( \psi \) satisfies the same conditions, then for \( \bar{r} \) in \( V \)

\[
\psi(\bar{r}) = \int_{V} (dV') \cdot G(\bar{r}, \bar{r}') S(\bar{r}').
\]

But suppose \( \psi \) satisfies inhomogeneous Dirichlet conditions on \( \partial V \):

\[
\psi(\bar{r}') \bigg|_{\bar{r}' \in \partial V} = \psi_0(\bar{r}') \quad \text{(specified)}
\]

Then we impose homogeneous Dirichlet conditions on \( G \):

\[
G(\bar{r}, \bar{r}') \bigg|_{\bar{r}' \in \partial V} = 0
\]

The first surface term is zero, but the 2nd contributes. For example, if \( S(\bar{r}) = 0 \) inside \( V \), \( \bar{r} \) inside \( V \)

\[
\psi(\bar{r}) = \int d\sigma' \cdot [\nabla' G(\bar{r}, \bar{r}')] \psi_0(\bar{r}')
\]

which expresses \( \psi \) in terms of its boundary values.
If \( \psi \) satisfies inhomogeneous Neumann conditions on \( \partial \Omega \)

\[
\frac{\partial \psi (\vec{r}')}{\partial n'} \bigg|_{\vec{r}' \in \partial \Omega} = N(\vec{r}') \text{ specified}
\]

then we use the Green's function which respects homogeneous Neumann condition

\[
\frac{\partial}{\partial n} \ G(\vec{r}, \vec{r}') \bigg|_{\vec{r}' \in \partial \Omega} = 0
\]

so, again if \( S = 0 \) inside \( \Omega \), \( (\vec{r} \text{ inside } \Omega) \)

\[
\psi(\vec{r}) = -\int d\sigma \ G(\vec{r}, \vec{r}') \ N(\vec{r}')
\]

\[
[\vec{n} \cdot \nabla = \frac{\partial}{\partial n}]
\]

[Finally if \( \psi \) satisfies inhomogeneous mixed B.C.,

\[
\frac{2}{\partial n}, \psi(\vec{r}') + \alpha(\vec{r}') \psi(\vec{r}') \bigg|_{\vec{r}' \in \partial \Omega} = F(\vec{r}')
\]

then when \( G \) satisfies homogeneous B.C. of the same type

\[
\frac{2}{\partial n} + \alpha(\vec{r}') \bigg|_{\vec{r}' \in \partial \Omega} = 0
\]