Connection between continued fractions and power series.

An example of a C.F. is

\[
C(x) = \frac{1}{1 + \frac{b_1 x}{1 + \frac{b_2 x}{1 + \frac{b_3 x}{1 + \frac{b_4 x}{1 + \ldots}}}}}
\]

For small enough \( x \) we should be able to expand this in a Taylor serie

\[
C(x) = \sum_{n=0}^{\infty} a_n x^n
\]

What is the connection between \( a_n \) and \( b_n \)?

By direct expansion we find

\[
\begin{align*}
a_0 &= 1 \\
a_1 &= -b_1 \\
a_2 &= b_1 (b_1 + b_2) \\
a_3 &= -b_1 \left[ b_1 b_3 + (b_1 + b_2)^2 \right] \\
a_4 &= b_1 \left[ b_1 (b_3 + b_4) + 2(b_1 + b_2) b_2 b_3 + (b_1 + b_2)^3 \right]
\end{align*}
\]

A nonlinear mapping from \( \{b_n\} \to \{a_n\} \)
Given the polynomials:

\[ P_0 = 1 \]
\[ P_1 = x \]
\[ P_{n+2} = x \cdot P_{n+1} + b_{n+1} \cdot P_n \]

Then \( P_2 = x^2 + b_1 \), \( P_3 = x^3 + (b_1 + b_1)x \), \( P_4 = x^4 + (b_1 + b_1)x^2 + b_1 b_1 \).

These are orthogonal with respect to a weight function \( w \):

\[ \int_{-a}^{a} w(x) P_n(x) P_m(x) \, dx = 0 \quad \text{if} \quad n \neq m \]

where

\[ \int_{-a}^{a} w^{-1}(x) x^n \, dx = a_n \quad \text{(Moments)} \]

\( w^{-1}(x) \) even

For

\[ \int_{-a}^{a} w^{-1}(x) P_1(x) P_0(x) \, dx = 0 \quad \text{parity} \]

\[ \int_{-a}^{a} w^{-1}(x) P_2(x) P_0(x) \, dx = \int_{-a}^{a} w^{-1}(x) (x^2 + b_1) \, dx \]

\[ = a_1 + b_1 a_0 = a_1 + b_1 \]

\[ \int_{-a}^{a} w^{-1}(x) P_3(x) P_1(x) \, dx = \int_{-a}^{a} w^{-1}(x) (x^3 + b_1 x + b_2) x \]

\[ = a_2 + b_1 a_1 + b_2 a_1 = a_2 - b_1 (b_1 + b_2) \]
\[ \begin{align*}
&= \int_a^b w(x) [x^4 + (b_1 + b_2 + b_3)x^2 + b_1 b_3] \\
&= a_2 + a_1 (b_1 + b_2 + b_3) + a_0 b_1 b_3 \\
&= b_1 (b_1 + b_2) - b_1 (b_1 + b_2 + b_3) + b_1 b_3 = 0.
\end{align*} \]

How are these orthogonal polynomials normalized?

\[ \int w^{-1} x^n = a_n = 1 \]
\[ \int w^{-1} x = a_1 = -b_1 \]
\[ \int w^{-1} x^2 = a_2 + 2a_1 b_1 + b_1^2 \]

\[ \begin{align*}
&= b_1 (b_1 + b_2) + 2(-b_1) b_1 + b_1^2 \\
&= b_1 b_2 \\
\end{align*} \]

\[ \int w^{-1} x^n = \prod_{i=1}^n (-1)^n b_i b_1 \cdots b_n \]
Proof: \[ \int dx w P_{n+2}^2 = \int dx w P_{n+2} (x P_{n+1} + b_{n+1} P_n) \]

\[ = \int dx w P_{n+1} (P_{n+3} - b_{n+2} P_{n+1}) \]

\[ = -b_{n+2} \int dx w P_{n+1}^2 \]

so if true for \( n \), true for \( n+1 \).

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Proof of orthogonality: Suppose \( \int w P_m (x) P_n (x) = 0 \)

and also \( \int w P_m^2 = (-1)^n b_n \), \( m \neq n \), \( m = 0, \ldots, N \)

\[ \int w P_m (x) P_{N+1} (x) = \int w P_m (x) (x P_N + b_N P_{N-1}) \]

\[ = b_N \int w P_m (x) P_{N-1} (x) \]

\[ + \frac{1}{b_m} \int w P_N (x) (P_{m+1} (x) - b_{m+1} P_m) \]

Zero by hyp. unless \( m = N-1 \)

\( (m = N \) int. vanishes by parity \). Then

\[ b_N \int w P_{N-1}^2 + \int w P_N^2 = b_N (-1)^{N-1} b_{N-1} \ldots b_1 \]

\[ + (-1)^N b_N \ldots b_1 \]

\[ = 0 \]