We shall then prove that the same formula holds for $k \rightarrow k + 1$, thereby establishing this formula, the Euler-Maclaurin summation formula, for all $k$. We proceed as follows. Note that
\[ B_{2k}(x) = \frac{B'_{2k+1}(x)}{2k+1} = \frac{B''_{2k+2}(x)}{(2k+1)(2k+2)}, \quad (4.32) \]
so that by integrating by parts, we rewrite the last term in Eq. (4.31) as
\[ \frac{1}{(2k)!} \int_0^1 f^{(2k)}(x) B_{2k+2}(x) dx = \frac{1}{(2k+2)!} \left[ f^{(2k)}(1) B'_{2k+2}(1) - f^{(2k)}(0) B'_{2k+2}(0) \right. \]
\[ - \int_0^1 f^{(2k+1)}(x) B'_{2k+2}(x) dx \]
\[ + \frac{1}{(2k+2)!} \left. \right] \left[ - f^{(2k+1)}(1) B_{2k+2}(1) + f^{(2k+1)}(0) B_{2k+2}(0) \right. \]
\[ + \int_0^1 f^{(2k+2)}(x) B_{2k+2}(x) dx \right], \quad (4.33) \]
where we have noted that for $k > 0$
\[ B'_{2k+2}(0) = (2k+2) B_{2k+1}(0) = 0, \quad (4.34a) \]
\[ B'_{2k+2}(1) = (2k+2) B_{2k+1}(1) = -(2k+2) B_{2k+1}(0) = 0. \quad (4.34b) \]

Hence
\[ \int_0^1 f(x) dx = \frac{1}{2} [f(1) + f(0)] \]
\[ - \sum_{m=1}^{k+1} \frac{B_{2m}}{(2m)!} \left[ f^{(2m-1)}(1) - f^{(2m-1)}(0) \right] \]
\[ + \frac{1}{(2k+2)!} \int_0^1 f^{(2k+2)}(x) B_{2k+2}(x) dx. \quad (4.35) \]
This is exactly Eq. (4.31) with $k$ replaced by $k + 1$; so since the formula is true for $k = 1$ it is true for all integers $k \geq 1$. Notice that the last term in this formula, the remainder, can also be written in the form
\[ - \frac{1}{(2k+3)!} \int_0^1 f^{(2k+3)}(x) B_{2k+3}(x) dx. \quad (4.36) \]
4.3. EULER-MACLAURIN SUMMATION FORMULA

Now consider the integral \((N\) a positive integer)

\[
\int_0^N f(s)\,ds = \sum_{k=0}^{N-1} \int_k^{k+1} f(s)\,ds = \sum_{k=0}^{N-1} \int_0^1 f(k + t)\,dt,
\]

(4.37)

where we have introduced a local variable \(t\). For the latter integral, we can use the Euler-Maclaurin sum formula, which here reads

\[
\int_0^1 f(k + t)\,dt = \frac{1}{2} [f(k + 1) + f(k)] - \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} \left[ f^{(2m-1)}(k + 1) - f^{(2m-1)}(k) \right] + \frac{1}{(2n)!} \int_0^1 f^{(2n)}(k + t)B_{2n}(t)\,dt.
\]

(4.38)

Now when we sum the first term here on the right-hand side over \(k\) we obtain

\[
\sum_{k=0}^{N-1} \frac{1}{2} [f(k + 1) + f(k)] = \sum_{k=0}^{N} f(k) - \frac{1}{2} [f(0) + f(N)],
\]

(4.39)

while the second term when summed on \(k\) involves

\[
\sum_{k=0}^{N-1} \left[ f^{(2m-1)}(k + 1) - f^{(2m-1)}(k) \right] = f^{(2m-1)}(N) - f^{(2m-1)}(0).
\]

(4.40)

Thus we find

\[
\int_0^N f(s)\,ds = \sum_{k=0}^{N} f(k) - \frac{1}{2} [f(0) + f(N)] - \sum_{m=1}^{n} \frac{1}{(2m)!} B_{2m} \left[ f^{(2m-1)}(N) - f^{(2m-1)}(0) \right] + \frac{1}{(2n)!} \int_0^1 \sum_{k=0}^{N-1} f^{(2n)}(t + k)B_{2n}(t)\,dt.
\]

(4.41)

Equivalently, we can write this as a relation between a finite sum and an integral, with a remainder \(R_n\):

\[
\sum_{n=0}^{N} f(k) = \int_0^N f(s)\,ds + \frac{1}{2} [f(0) + f(N)] + \sum_{m=1}^{n} \frac{1}{(2m)!} B_{2m} \left[ f^{(2m-1)}(N) - f^{(2m-1)}(0) \right] + R_n.
\]

(4.42)
where the remainder

$$R_n = - \frac{1}{(2n)!} \int_0^1 \sum_{k=0}^{N-1} f^{(2n)}(t+k)B_{2n}(t) \, dt. \quad (4.43)$$

is often assumed to vanish as $n \to \infty$. Note that the remainder can also be written as

$$R_n = - \frac{1}{(2n)!} \int_0^N f^{(2n)}(t)B_{2n}(t-[t]) \, dt, \quad (4.44)$$

where $[t]$ signifies the greatest integer less than or equal to $t$.

### 4.3.1 Examples

1. Use the Euler-Maclaurin formula to evaluate the sum $\sum_{n=0}^{N} \cos(2\pi n/N)$.

$$\sum_{n=0}^{N} \cos \frac{2\pi n}{N} = \int_0^N dn \cos \frac{2\pi n}{N} + \frac{1}{2}(1+1) + 0 = 1, \quad (4.45)$$

because

$$f^{(2m-1)}(0) = f^{(2m-1)}(N) = 0 \quad (4.46)$$

and

$$\int_0^N dn \cos \frac{2\pi n}{N} = \frac{N}{2\pi} \int_0^{2\pi} dx \cos x = 0. \quad (4.47)$$

Of course, the sum may be carried out directly,

$$\sum_{n=0}^{N} \cos \frac{2\pi n}{N} = \frac{1}{2} \sum_{0}^{N} \left( e^{i2\pi n/N} + e^{-i2\pi n/N} \right)$$

$$= \frac{1}{2} \left[ \frac{1 - e^{2\pi i(N+1)/N}}{1 - e^{2\pi i/N}} + \frac{1 - e^{-2\pi i(N+1)/N}}{1 - e^{-2\pi i/N}} \right]$$

$$= \frac{1}{2} (1+1) = 1. \quad (4.48)$$

2. The following sum occurs, for example, in computing the vacuum energy in a cosmological model:

$$\sum_{l=0}^{\infty} (2l+1)e^{-l(l+1)t}. \quad (4.49)$$

How does this behave as $t \to 0$? We will answer this question by using the Euler-Maclaurin formula assuming that the remainder $R_n$ tends to zero as $n \to \infty$. Thus we will write the limiting form of that sum formula as

$$\sum_{l=0}^{\infty} f(l) = \int_0^\infty dl \, f(l) + \frac{1}{2} [f(\infty) + f(0)]$$

$$+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left[ f^{(2k-1)}(\infty) - f^{(2k-1)}(0) \right]. \quad (4.50)$$
4.3. Euler-Maclaurin Summation Formula

Here

\[ f(l) = (2l + 1)e^{-l(l+1)t}, \]  

so that

\[ f(\infty) = f^{(2k-1)}(\infty) = 0, \]  

while a very simple calculation shows

\begin{align*}
  f(0) &= 1, \quad \text{(4.53a)} \\
  f'(0) &= 2 - t, \quad \text{(4.53b)} \\
  f''(0) &= -12t + O(t^2), \quad \text{(4.53c)} \\
  f^{(2k-1)}(0) &= O(t^2), \quad k \geq 3. \quad \text{(4.53d)}
\end{align*}

Thus Eq. (4.50) yields

\[
\sum_{l=0}^{\infty} (2l + 1)e^{-l(l+1)t} = \int_0^{\infty} dl \, (2l + 1)e^{-l(l+1)t} + \frac{1}{2} \\
- \frac{B_2}{2} f'(0) - \frac{B_4}{4!} f''(0) - \ldots \\
= \frac{1}{l} \int_0^{\infty} du \, e^{-u} + \frac{1}{2} + \frac{1}{2} \left( \frac{1}{6} \right) (t - 2) \\
+ \frac{1}{4!} \left( -\frac{1}{30} \right) [12t + O(t^2)] + O(t^2) \\
= \frac{1}{l} + \frac{1}{3} + \frac{t}{15} + \frac{4}{315} t^2 + \frac{1}{315} t^3 + \ldots \tag{4.54}
\]

Here the integral was evaluated by making the substitution \( u = l(l+1)t \), \( du = (2l + 1)t \, dl \), and in the last line we have displayed the next two terms in this asymptotic expansion for small \( t \).