Homework Assignment # 2  

Due: Monday, Jan. 31st  

Instructions:  

Homework is due at the start of the Monday’s class. You may turn it in at the lab, in my mailbox office, or after class on Monday.  

Reading: My presentation is taken from Callen and from Weinreich. Garrod’s presentation is the same since it is based on Callen.  
Please read Reichl, Chapter 3 on Phase transitions. We’ll have a reading quiz on it sometime next week.  

Problems: Please solve the following:  

1. For the allowable fundamental equations in the last homework assignment, derive an expression for the free energy, $F$.  

2. Consider a thermodynamic system where the differential of the internal energy is given by:  
   \[ dU = T \, dS + f \, dx \]  
   (1)  
where $f$ is an intensive force variable and $x$ is an extensive displacement variable.  
(a) Calculate all the Maxwell relations derivable from the second partials of $U$.  
(b) Calculate the following derivatives in terms of the natural variables of the system:  
   i. $(df/dU)_T$  
   ii. $(dU/dS)_f$  
   iii. $(dx/df)_S$  
(c) Prove the following identities:  
   i.  
   \[ \left( \frac{dU}{dS} \right)_f + f \left( \frac{dT}{df} \right)_S = T \]  
   ii.  
   \[ \left( \frac{df}{dx} \right)_T = \left( \frac{df}{dx} \right)_S \left( \frac{df}{dT} \right)_S \left( \frac{dT}{dS} \right)_S \]  

3. In classical mechanics we extremize the action  
   \[ A[q(t), \dot{q}(t)] = \int \mathcal{L}(q(t), \dot{q}(t)) \, dt \]  
where $\mathcal{L}$ is the Lagrangian:  
\[ \mathcal{L}(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - V(q) \]
which we extremize as a function of \( q(t) \). (That is, the functions \( q(t) \) and \( \dot{q}(t) \) are an infinite number of variables, and we extremize \( \mathcal{L} \) with respect to the entire set.) In doing so we obtain the standard Euler-Lagrange equation:

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0
\]

(a) Rather than chose \((q(t), \dot{q}(t))\) as our variables, let’s assume we’d rather work with \((q(t), p(t))\), where

\[
p(t) = \frac{\partial \mathcal{L}}{\partial \dot{q}}
\]

Construct a new function which contains all of the same information as \( \mathcal{L} \).

(b) Take the differential of this new function, and use it to determine the partial derivatives of the new function with respect to \( q \) and \( p \).

(c) Repeat this process for the Lagrangian:

\[
L = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) + \alpha \left( x \dot{y} - y \dot{x} \right)
\]

where \( x \) and \( y \) are the standard Cartesian coordinates, and a “dot” denotes a time derivative i.e. \( \dot{x} \equiv dx/dt \). The variable \( m \) is the mass, and \( \alpha \) is a positive constant. Explicitly derive the Hamiltonian for this system.

4. In class we performed a Legendre transformation on the internal energy \( U(S, V, N) \) to determine the free energy \( F(T, V, N) \). However, we know that rather than minimize the energy, we can maximize the entropy, \( S(U, V, N) \).

(a) Determine the free entropy, \( S^r(T, V, N) \), by performing a Legendre transform on \( S(U, V, N) \). (Such Legendre transformed entropies are called Massieu functions.) Derive an expression for \( dS^r \) in terms of its basic thermodynamic variables.

(b) Is \( S^r \) minimized or maximized in equilibrium, for fixed \( T, V, \) and \( N \)? Prove your answer.