1). A particle of mass $m$ starts at rest on top of a smooth fixed hemisphere of radius $a$.

(a) Find the constraint equation $f(r, \theta) = 0$.

(b) Find the Lagrangian in terms of polar coordinates $r$ and $\theta$.

(c) Determine the Euler-Lagrange equation with a Lagrange multiplier $\lambda$.

(d) Find the Lagrange multiplier $\lambda$ and the force of constraint.

(e) Determine the angle at which the particle leaves the hemisphere.

2). Find the condition for stable circular orbits for a potential energy of the following form

$$V(r) = -\frac{c}{r^\lambda}$$

where $\lambda < 2$, $c \cdot \lambda > 0$, and $\lambda \neq 0$. Show that the angular frequency for small radial oscillations $\omega_r$ is related to the orbit angular frequency $\omega_\theta$ by

$$\omega_r = \omega_\theta \sqrt{2 - \lambda}.$$

This result implies that the orbit is closed and the motion is periodic only if $\sqrt{2 - \lambda}$ is a rational number. Sketch the orbits for $\lambda = 1$ (Coulomb potential energy) and $\lambda = -2$ (harmonic oscillator).
3). A particle of mass $m$ moves on a plane under the influence of a central force

$$\vec{F} = -c^2 \frac{\hat{r}}{r^{5/2}}.$$ 

where $\hat{r}$ is the unit vector in the radial direction.

(a) Calculate the potential energy $U(r)$.

(b) By means of the effective potential energy discuss the motion.

(c) Find the radius of any circular orbit in terms of the angular momentum and calculate the period for the orbit.

(d) Determine the frequency for small radial oscillations about the circular orbit of part (c).

4). Let us consider the motion of a particle in the central force

$$\vec{F} = -kr.$$ 

Use Cartesian coordinates and show that the orbit is an ellipse with the force center at the center of the ellipse.