Problem (1)

Let us consider the following operators on a Hilbert space $V^3$:

$$L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

(i) Consider the state

$$|\psi\rangle = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}$$

in the $L_z$ basis. If $L_z^2$ is measured in this state and a result +1 is obtained, what is the state after the measurement? How probable was this result? If $L_z$ is measured, what are the outcomes and respective probabilities?

(ii) A particle is in a state for which the probabilities are

$$P(\ell_z = 1) = 1/4,$$
$$P(\ell_z = 0) = 1/2, \text{ and}$$
$$P(\ell_z = -1) = 1/4.$$

The most general normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2} |\ell_z = 1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}} |\ell_z = 0\rangle + \frac{e^{i\delta_3}}{2} |\ell_z = -1\rangle.$$ 

If $|\psi\rangle$ is a normalized state then the state $e^{i\theta} |\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factors $e^{i\delta_i}$ are irrelevant? Calculate $P(\ell_x = 1)$, $P(\ell_x = 0)$, and $P(\ell_x = -1)$, for

(a) $\delta_1 = \delta_2 = \delta_3 = 0$,

(b) $\delta_1 = \delta_3 = 0, \text{ and } \delta_2 = \pi$,

(c) $|\psi'\rangle = e^{-i\delta_1} |\psi\rangle$. 

Problem (2)

For a free particle in one dimension, the Schrödinger Equation is

$$i\hbar \frac{d}{dt} |\Psi\rangle = H |\Psi\rangle \quad \text{with} \quad H = \frac{P^2}{2m}.$$ 

(a) Show that the wave function $\phi(p)$ in the momentum space or the $p-$basis is

$$\phi(p) = \langle p | \Psi \rangle = Ne^{-\frac{(i/\hbar)(\frac{p^2}{2m})t}}$$

where $N$ is the normalization constant.

(b) Find the wave function $\Psi(x,t) = \langle x | \Psi \rangle$ in the coordinate space or the $x-$basis by applying the Fourier transform

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \phi(\hbar k) dk$$

where $\phi(p) = \phi(\hbar k)$ with $k = p/\hbar$.

Problem (3)

For a particle moving in an infinite square well of width $2a$ with the potential energy

$$V(x) = \begin{cases} 
0, & \text{for } -a \leq x \leq a \text{ with } a > 0, \text{ and} \\
\infty, & \text{otherwise,} 
\end{cases}$$

its normalized wave function inside the well at time $t = 0$ is

$$\Psi(x,0) = C \left[ \sin \frac{\pi x}{a} + \frac{1}{4} \cos \frac{3\pi x}{2a} \right]$$

and $\Psi(x,0) = 0$ for $x^2 \geq a^2$.

(a) Calculate the coefficient $C$.

(b) What is the wave function $\Psi(x,t)$?

(c) If a measurement of total energy is made, what are the possible results of such a measurement, and what is the probability to measure each of them?

(d) What is the expectation value of the energy $\langle E \rangle$?
Problem (4)

Let us consider a one-dimensional quantum harmonic oscillator with the Hamiltonian

\[ H = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 X^2, \quad \text{and} \]

\[ H |n \rangle = E_n |n \rangle. \]

(a) Find \( \langle X \rangle, \langle P \rangle, \langle X^2 \rangle, \langle P^2 \rangle \) and \( \Delta X \Delta P \) in the state \( |n \rangle \).

(b) What is the uncertainty relation for the ground state.