Problem (1): Normalization of Wave Functions

(a) Find the normalization constant $A$ for

$$\psi(x) = A \exp[-\frac{1}{2c\hbar}(x - \langle X \rangle)^2 + \frac{i}{\hbar}\langle P \rangle x]$$

such that

$$\int |\psi(x)|^2 dx = 1$$

where $c$ is a constant.

(b) Calculate the Fourier transform $g(\vec{k})$ of the function

$$f(\vec{x}) = Ne^{-r^2/a^2}e^{-\mu r}, \quad r = |\vec{x}|$$

in the 3-dimensional space. Find the normalization constant $N$ for $f(\vec{r})$ and an exact expression of $g(\vec{k})$ for $\mu = 0$. For $\mu \neq 0$, express $g(\vec{k})$ as an integral.

Problem (2): Expectation Value and Uncertainty

A particle is represented at time $t = 0$ by the following wave function

$$\psi(x) = \begin{cases} A(a^2 - x^2) & \text{for } -a \leq x \leq a, \quad a > 0, \text{and} \\ 0, & \text{for } x^2 > a^2, \end{cases}$$

(a) Determine the normalization constant $A$.

(b) What is the expectation value of $X$ at time $t = 0$?

(c) What is the expectation value of $P$ at time $t = 0$?

(d) Find the the expectation value of $X^2$.

(e) Find the the expectation value of $P^2$.

(f) Find the uncertainty ($\Delta X$) in $X$. 
Problem (3): Ehrenfest’s Theorem

(a) Find $d\langle P \rangle / dt$ by applying the Ehrenfest’s Theorem. Then compare your result with Hamilton’s equation for $dP/dt$.

(b) Apply the Ehrenfest’s Theorem for a quantum operator $\Omega$

$$i\hbar \frac{d}{dt} \langle \Omega \rangle = \langle \psi | [\Omega, H] | \psi \rangle$$

to show that

$$\frac{d}{dt} \langle XP \rangle = 2\langle T \rangle - \langle X \frac{dV}{dx} \rangle,$$

where $T$ is the kinetic energy and the Hamiltonian $H = T + V$. In a stationary state, the left hand side is zero so

$$2\langle T \rangle = \langle X \frac{dV}{dx} \rangle.$$

Problem (4): Schrödinger Equation in One Dimension

Let us consider a particle bound in a delta function potential in one dimension

$$V(x) = -\alpha \delta(x), \quad \alpha > 0,$$

where $\alpha$ is a constant.

(a) Find the wave function that satisfies the equation of motion

$$H \psi(x) = E \psi(x)$$

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

in terms of $k = (-2mE/h^2)^{1/2}$, where $E$ is the eigenvalue of $H$ and $\psi(x)$ is continuous at $x = 0$.

(b) Calculate the average total energy $\langle E \rangle$, the average potential energy $\langle V \rangle$ and then the average kinetic energy $\langle T \rangle$ of this particle with the relation $\langle T \rangle = \langle E \rangle - \langle V \rangle$.

(c) Calculate $\langle T \rangle$ for the previous problem directly as

$$\int dx \psi^*(x)(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2})\psi(x)$$

and compare with the previous result.