Problem (1)

If $A$ and $B$ are two operators satisfying $[[A, B], A] = 0$, apply Mathematical Induction and show that the relation

$$[A^m, B] = mA^{m-1}[A, B]$$

holds for all positive integers $m$.

Problem (2)

Express the Fourier transforms $[G(k)]$ of the following functions $[F(x)]$ in terms of $g(k)$ that is the Fourier transform of $f(x)$.

(a) $F(x) = f(x + a)$

(b) $F(x) = f^*(x)$

(c) $F(x) = f(-x)$

(d) $F(x) = e^{i\mu x}f(x)$ where $\mu$ is real.

(e) $F(x) = df(x)/dx$

Problem (3)

Apply two dimensional polar coordinates and Show that

$$I(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}},$$

which is the Gaussian integral. Then find the Fourier transform of $f(x) = e^{-\alpha x^2/2}$.

Problem (4)

Let us consider the following operators on a Hilbert space $V^3$:

$$L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Find the eigenvalues $\ell_z$ and normalized eigenvectors $|\ell_z\rangle$ of $L_z$.

(b) Take the state in which $\ell_z = 1$. In this state, what are $\langle L_x \rangle, \langle L_z^2 \rangle$, and $\Delta L_x$.

(c) Find the eigenvalues and the normalized eigenvectors of $L_x$ in $L_z$ basis.

(d) If the particle is in the state with $\ell_z = -1$, and $L_x$ is measured, what are the possible outcomes and their probabilities?