1) The fine structure Hamiltonian \( (H_{fs}) \) for the Hydrogen atom can be written as:

\[
H_{fs} = H_{\text{kin}} + H_{so} + H_{D}
\]

where

\[
H_{\text{kin}} = -\frac{1}{2m^2c^2}P^2
\]
\[
H_{so} = \frac{1}{2m^2c^2} \frac{1}{R} \frac{dV(r)}{dr} \mathbf{L} \cdot \mathbf{S}
\]
\[
H_{D} = \frac{\pi e^2 \hbar^2}{2m^2c^2} \delta^3(R)
\]

In class I gave the results for the \( n=2 \) state of Hydrogen. Determine the fine structure corrections for the \( n=2 \) state of Hydrogen. (Show all work)

2) The energy shift due to the relativistic correction and the spin-orbit coupling is:

\[
E_{fs} = \frac{E_n^2}{2mc^2} \left(3 - \frac{4n}{j + 0.5}\right)
\]

Derive this equation. Note that \( j = \ell \pm 1/2 \). Treat the plus sign and minus sign separately, and you will get the same answer either way.

Hints:

1) Use the virial theorem to determine \( \langle \frac{1}{r} \rangle \).
2) To determine \( \langle \frac{1}{r^2} \rangle \) use the relationship

\[
\frac{\partial E_n}{\partial \lambda} = \langle \Psi | \frac{\partial H}{\partial \lambda} | \Psi \rangle
\]

3) \( \langle \frac{1}{r^{1/2}} \rangle = \frac{1}{(\ell + 0.5)(\ell + 1)\sqrt{\gamma}} \)

3) One can combine the result of problem 2 with the unperturbed energy to obtain the energy levels of hydrogen, including fine structure:

\[
E_{nj} = -\frac{13.6eV}{n^2} \left[1 + \alpha^2 \left(\frac{n}{j + 0.5} - \frac{3}{4}\right)\right]^{-1/2}
\]

The exact fine-structure formula for hydrogen (obtained from the Dirac equation without using perturbation theory is:

\[
E_{nj} = nc^2 \left[1 + \left(\frac{\alpha}{n - (j + 0.5) + \sqrt{(j + 0.5)^2 - \alpha^2}}\right)^2\right]^{0.5} - 1
\]

Expand this to order \( \alpha^4 \) and show that you recover the same answer as in perturbation theory.