An Example

We consider a circular cylinder of radius $a$ and discuss the mode of lowest eigenvalue with $m = 0$:

$$
\varphi_{01}(\rho, \phi) = \frac{1}{(2\pi)^{1/2}} P_{01}(\rho), \quad P_{01}(a) = 0.
$$

(3.1)

The equation defining the successive iterations is (henceforth the subscripts are omitted)

$$
- \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} P^{(n+1)}(\rho) = P^{(n)}(\rho),
$$

(3.2)

and the required integrals are

$$
[m + n] = \int_{0}^{a} d\rho \rho P^{(m)}(\rho) P^{(n)}(\rho).
$$

(3.3)

Inasmuch as $a$ is the natural unit of length, and the various operations of differentiation and integration are even in $\rho$, it is expedient to introduce the variable

$$
t = (\rho/a)^2,
$$

(3.4)

and define the functions

$$
T^{(n)}(t) = \left(\frac{4}{a^2}\right)^n P^{(n)}(\rho),
$$

(3.5)

which obey the boundary condition

$$
T^{(n)}(1) = 0,
$$

(3.6)

and the differential equation

$$
- \frac{d}{dt} \frac{d}{dt} T^{(n+1)}(t) = T^{(n)}(t).
$$

(3.7)
The integrals (3.3) now appear as

\[ [m + n] = 2 \left( \frac{a^2}{4} \right)^{m+n+1} \langle m + n \rangle, \tag{3.8} \]

with

\[ \langle m + n \rangle = \int_0^1 dt \ T^{(m)}(t) T^{(n)}(t). \tag{3.9} \]

In this notation, the \( n \)th approximation to the desired eigenvalue is given by

\[ \lambda^{(n)} = \frac{[2n - 1]}{[2n]} = \frac{4}{a^2} \frac{(2n - 1)}{(2n)}, \tag{3.10} \]

which is valid both for integer and integer + \( \frac{1}{2} \) values of \( n \).

We shall find it desirable to introduce a pre-iteration function, \( T^{(-1)}(t) \), according to

\[ -\frac{d}{dt} t \frac{d}{dt} T^{(0)}(t) = T^{(-1)}(t), \tag{3.11} \]

which function does not satisfy the boundary condition. Indeed, we use

\[ T^{(-1)}(t) = 1, \tag{3.12} \]

for that makes it the beginning of the line:

\[ -\frac{d}{dt} t \frac{d}{dt} T^{(-1)}(t) = T^{(-2)}(t) = 0. \tag{3.13} \]

To see what benefits emerge thereby, consider

\[ \langle k \rangle = \int_0^1 dt \ T^{(k)} T^{(0)} = \int_0^1 dt \left[ -\frac{d}{dt} t \frac{d}{dt} T^{(k+1)} \right] T^{(0)} = \int_0^1 dt \ T^{(k+1)} T^{(-1)}, \tag{3.14} \]

for both \( T^{(k+1)} \) and \( T^{(0)} \) obey the boundary condition. Then, with the choice (3.12), we reach the simple evaluation

\[ \langle k \rangle = \int_0^1 dt \left[ -\frac{d}{dt} t \frac{d}{dt} T^{(k+2)} \right] = -\frac{d}{dt} T^{(k+2)}(1). \tag{3.15} \]

In carrying out the integrations required to produce the successive \( T^{(n)}(t) \), it suffices to note that

\[ -\frac{d}{dt} t \frac{d}{dt} \left[ -\frac{t^{n+1}}{(n + 1)^2} \right] = t^n. \tag{3.16} \]
Thus, the solution of (3.11) that obeys the boundary condition (3.6) is (here \( n = 0 \))
\[
T^{(0)}(t) = 1 - t = T^{(-1)}(t) - t .
\] (3.17)

Then, we get in succession,
\[
T^{(1)} = T^{(0)} + \frac{1}{2^2} \left( t^2 - T^{(-1)} \right) ,
\]
(3.18a)
\[
T^{(2)} = T^{(1)} - \frac{1}{2^2} T^{(0)} - \frac{1}{2^2 3^2} \left( t^3 - T^{(-1)} \right) ,
\]
(3.18b)
\[
T^{(3)} = T^{(2)} - \frac{1}{(2!)^2} T^{(1)} + \frac{1}{(3!)^2} T^{(0)} + \frac{1}{(4!)^2} \left( t^4 - T^{(-1)} \right) ,
\]
(3.18c)
and, in general,
\[
T^{(n)} = \sum_{l=1}^{n+1} \frac{(-1)^{l-1}}{(l!)^2} T^{(n-l)} + (-1)^{n-1} \frac{n+1}{(n+1)!} \frac{t^{n+1}}{(n+1)!} .
\] (3.19)

From this we deduce recurrence relations for the integrals as evaluated in (3.15),
\[
\langle k \rangle = \sum_{l=1}^{k+3} \frac{(-1)^{l-1}}{(l!)^2} \langle k - l \rangle + (-1)^k \frac{1}{(k+3)!} .
\] (3.20)

The quantities appearing here with negative numbers, \( \langle -3 \rangle , \langle -2 \rangle , \langle -1 \rangle \), are to be understood in the sense of the last version of (3.14), and therefore are integrals of products with one factor of \( T^{(-1)}(t) \),
\[
\langle -3 \rangle = \int_0^1 dt T^{(-2)} T^{(-1)} = 0 ,
\] (3.21a)
\[
\langle -2 \rangle = \int_0^1 dt T^{(-1)} T^{(-1)} = 1 ,
\] (3.21b)
\[
\langle -1 \rangle = \int_0^1 dt T^{(0)} T^{(-1)} = \frac{1}{2} ,
\] (3.21c)
and then we get
\[
\langle 0 \rangle = \frac{1}{2} - \frac{1}{(2!)^2} + \frac{1}{2! 3!} = \frac{1}{3} ,
\] (3.22a)
\[
\langle 1 \rangle = \frac{1}{3} - \frac{1}{(2!)^2} + \frac{1}{2! 3! 4!} = \frac{11}{48} ,
\] (3.22b)
\[
\langle 2 \rangle = \frac{11}{48} - \frac{1}{(2!)^2} \frac{1}{3} + \frac{1}{(3!)^2} 2 - \frac{1}{(4!)^2} + \frac{1}{4! 5!} = \frac{19}{120} ,
\] (3.22c)
\[
\langle 3 \rangle = \frac{19}{120} - \frac{1}{(2!)^2} \frac{1}{48} + \frac{1}{(3!)^2} 3 - \frac{1}{(4!)^2} 2 + \frac{1}{(5!)^2} 6 - \frac{1}{5! 6!} = \frac{473}{4320} ,
\] (3.22d)
\[
\langle 4 \rangle = \frac{473}{4320} - \frac{1}{(2!)^2} \frac{1}{120} + \frac{1}{(3!)^2} \frac{1}{48} - \frac{1}{(4!)^2} 3 + \frac{1}{(5!)^2} 2 - \frac{1}{(6!)^2} + \frac{1}{6! 7!}
\].
\[ \langle 3 \rangle = \sum_{n=0}^{\infty} P_{on}(\rho)P_{on}(\rho') = \log \frac{a}{\rho'}, \] (3.24)

\[ \gamma_{mn} = a^{-1} \langle mn \rangle, \]

In order to produce corresponding lower limits, we turn our attention to \( \lambda_2 \). The best result for \( \lambda_2 a^2 \) so far is [(CE25.75)]\(^1\) 19.74. Now we apply the technique developed in Chapter 23 of Ref. [1], beginning with the use made of (CE23.33). We first state, and then derive the analogue of that equation for the \( m = 0 \) modes of the circle:
Looking back at Chapter 21 of Ref. [1], we recognize here, for \( m = 0 \), the \( k \to 0 \) limit of the relation [cf. (CE19.103) and (CE19.73)]

\[
\sum_{n=1}^{\infty} \frac{P_{mn}(\rho)P_{mn}(\rho')}{k^2 + \gamma^2_{mn}} = I_m(k\rho) - I_m(k\rho') \frac{K_m(ka)}{I_m(ka)}. \tag{3.25}
\]

Now according to (CE18.25) and (CE18.66), we have

\[
t \ll 1 : \quad K_0(t) \sim \log \frac{1}{t} + \text{constant}, \quad I_0(t) \sim 1, \tag{3.26}
\]

from which (3.24) follows.

This verification can, of course, be performed more directly. The left-hand side of (3.24) – as in (CE19.103), call it \( g_0(\rho, \rho') \) – obeys

\[
-\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} g_0(\rho, \rho') = \frac{1}{\rho} \delta(\rho - \rho'), \tag{3.27a}
\]

\[
g_0(a, \rho') = 0. \tag{3.27b}
\]

Then we see that \( \log(a/\rho) \) vanishes for \( \rho = a \), and that

\[
-\rho \frac{\partial}{\partial \rho} \log \frac{a}{\rho} = \begin{cases} 
1, & \rho > \rho' \\
0, & \rho < \rho' 
\end{cases}, \tag{3.28}
\]

the derivative of this discontinuous function produces the required delta function.

Now we follow the example of (CE23.39) and derive, by integrating (3.24) with \( \rho = \rho' \)

\[
\sum_{n=1}^{\infty} \frac{1}{\gamma_{0n}^2} = \int_0^a d\rho \rho \log \frac{a}{\rho} = \frac{a^2}{4} \int_0^1 dt \log \frac{1}{t} = \frac{a^2}{4}, \tag{3.29}
\]

in which we have introduced the variable (3.4). As we remarked in (CE25.51), such a relation provides a lower limit to \( \lambda_2 \) in terms of an upper limit to \( \lambda_1 \):

\[
\frac{1}{\lambda_1 a^2} + \frac{1}{2 \lambda_2 a^2} = \frac{1}{4}. \tag{3.30}
\]

If we use the best upper limit in (3.23f), that is for \( n = 5/2 \), we get \( \lambda_2 a^2 = 12.97 \), which is a very poor result. Accordingly, we try the next stage, the analogue of (CE23.44), which involves integrating the square of (3.24):

\[
\sum_{n=1}^{\infty} \frac{1}{\gamma_{0n}^2} = \frac{1}{8} \int_0^a d\rho \rho \int_0^\rho d\rho' \left( \log \frac{a}{\rho} \right)^2 \\
= \frac{a^4}{8} \int_0^1 dt \int_0^t dt' \left( \log \frac{1}{t} \right)^2 \\
= \frac{a^4}{8} \int_0^1 dt t \left( \log \frac{1}{t} \right)^2 = \frac{a^4}{32}. \tag{3.31}
\]
the latter integral, like that of (3.29), being performed by partial integration, or by a change of variable: \( t = \exp(-x) \). Now we get

\[
\frac{1}{(\lambda_1 a^2)^2} + \frac{1}{(\lambda_2 a^2)^2} = \frac{1}{32}, \tag{3.32}
\]

from which we get

\[
\lambda_2 a^2 = 27.208, \tag{3.33}
\]

considerably better than the 19.74 produced by comparison with a square.

But before we examine how well (3.33) performs, let us see what we can learn from (3.25) by putting \( \rho = \rho' \) and integrating, namely:

\[
\sum_{n=1}^{\infty} \frac{1}{k^2 + \gamma^2_{mn}} = \int_{0}^{a} d\rho \rho I_m(k\rho)K_m(k\rho), \tag{3.34}
\]

where

\[
K_m(t) = K_m(t) - I_m(t) \frac{K_m(ka)}{I_m(ka)}. \tag{3.35}
\]

The two functions that enter the integral of (3.34) obey the same differential equation, for \( \rho > 0 \),

\[
\left[ \frac{1}{\rho} \frac{d}{d\rho} \left( \frac{1}{\rho} \frac{d}{d\rho} \right) - k^2 - \frac{m^2}{\rho^2} \right] \left\{ I_m(k\rho) \frac{K_m(k\rho)}{K_m(ka)} \right\} = 0. \tag{3.36}
\]

We also need the differential equations obeyed by

\[
\frac{\partial}{\partial k} I_m(k\rho) = \rho I_m'(k\rho), \tag{3.37}
\]

it is

\[
\left[ \frac{1}{\rho} \frac{d}{d\rho} \left( \frac{1}{\rho} \frac{d}{d\rho} \right) - k^2 - \frac{m^2}{\rho^2} \right] \rho I_m'(k\rho) = 2kI_m(k\rho). \tag{3.38}
\]

Then cross multiplication between the latter equation and that for \( K_m \), in the manner of (CE19.95), yields

\[
\frac{d}{d\rho} \left[ K_m(k\rho) \rho \frac{d}{d\rho} \rho I_m'(k\rho) - \rho I_m'(k\rho) \rho \frac{d}{d\rho} K_m(k\rho) \right] = 2k\rho I_m(k\rho)K_m(k\rho), \tag{3.39}
\]

so that the integrand of (3.34) is a total differential.

At the upper limit of the integral, \( \rho = a \),

\[
K_m(ka) = 0, \tag{3.40}
\]

and

\[
\rho \frac{d}{d\rho} K_m(k\rho) \bigg|_{\rho=a} = ka \left[ K_m' - I_m' \frac{K_m}{I_m} \right](ka) = -\frac{1}{I_m(ka)}, \tag{3.41}
\]
according to the Wronskian (CE19.85). To handle the lower limit, $\rho = 0$, we recognize that the structure being differentiated in (3.39) can, apart from a factor of $1/k$, be presented as

\[ K_m(t) \left( t \frac{d}{dt} \right)^2 I_m(t) - t \frac{d}{dt} I_m(t) t \frac{d}{dt} K_m(t), \quad (3.42) \]

for it is only through the singularity of $K_m(t)$ at $t = 0$ that a finite contribution can emerge. Now, for small values of $t$, $I_m(t) \sim t^m$, \[(CE18.66)\] and

\[ t \ll 1 : \quad t \frac{d}{dt} I_m(t) \sim mI_m(t), \quad (3.43) \]

so that (3.42) becomes

\[ mt [K_m(t)I'_m(t) - I_m(t)K'_m(t)] = m, \quad (3.44) \]

which again employs the Wronskian. Thus the integral in (3.34) equals

\[ \frac{1}{2k} \left[ \frac{I'_m(ka)}{I_m(ka)} - \frac{m}{k} \right]. \quad (3.45) \]

The result, presented as

\[ \frac{I'_m(t)}{I_m(t)} = \frac{m}{t} + 2t \sum_{n=1}^{\infty} \frac{1}{t^2 + (\gamma_{mn}a)^2}, \quad (3.46) \]

is not unfamiliar; we have derived the pole expansion of the logarithmic derivative of $I_m(t)$ in terms of the behavior at $t = 0$ and the imaginary roots at $\pm i\gamma_{mn}$ [cf. (23.46)]. Then, the initial terms of the power series expansion

\[ m \geq 0 : \quad I_m(t) = \left( \frac{1}{2} \right)^{m+1} m! \left[ 1 + \frac{1}{2} \left( \frac{1}{2} \right)^2 + \frac{1}{2} \left( \frac{1}{2} \right)^4 + \cdots \right], \quad (3.47) \]

with its logarithmic consequence

\[ \frac{I'_m(t)}{I_m(t)} = \frac{m}{t} + \frac{1}{2} \frac{t}{m+1} - \frac{1}{8} \frac{t^3}{(m+1)(m+2)} + \cdots, \quad (3.48) \]

give the first two of an infinite set of summations as

\[ \sum_{n=1}^{\infty} \frac{1}{\lambda_{mn}a^2} = \frac{1}{4} \frac{1}{m+1}, \quad (3.49a) \]

\[ \sum_{n=1}^{\infty} \frac{1}{(\lambda_{mn}a^2)^2} = \frac{1}{16} \frac{1}{(m+1)^2(m+2)}. \quad (3.49b) \]
The results already obtained in (3.29) and (3.31) for \( m = 0 \) are repeated, and, with the substitution \( m \to l + \frac{1}{2} \), we regain the spherical Bessel function summations of (23.40) and (23.44).

We record, for convenience, the consecutive differences, \( \lambda^{(n)}a^2 - \lambda^{(n+1/2)}a^2 \), with \( n = 0, 1/2, \ldots, 2 \), and the ratios of adjacent differences,

\[
\frac{\lambda^{(n)}a^2 - \lambda^{(n+1/2)}a^2}{\lambda^{(n+1/2)}a^2 - \lambda^{(n+1)}a^2},
\]

as found in (3.23a)–(3.23f):

\[
\begin{align*}
\lambda^{(0)}a^2 - \lambda^{(1/2)}a^2 &= 0.1818182 \\
\lambda^{(1/2)}a^2 - \lambda^{(1)}a^2 &= 0.0287081 \\
\lambda^{(1)}a^2 - \lambda^{(3/2)}a^2 &= 0.0051185 \\
\lambda^{(3/2)}a^2 - \lambda^{(2)}a^2 &= 0.0009491 \\
\lambda^{(2)}a^2 - \lambda^{(5/2)}a^2 &= 0.001785 \\
\lambda^{(5/2)}a^2 - \lambda^{(3)}a^2 &= 0.000338 \\
\lambda^{(3)}a^2 - \lambda^{(7/2)}a^2 &= 0.000064
\end{align*}
\]

Now let us adopt, provisionally, the lower limit of \( \lambda_2 \) given in (3.33). The following exhibits the lower bounds [(25.107)] thereby produced for \( n = \frac{1}{2}, \ldots, \frac{5}{2} \), along with the corresponding upper bounds:

\[
\begin{align*}
n = \frac{1}{2} : & \quad 5.8181818 > \lambda_1a^2 > 5.7687260 \\
n = 1 : & \quad 5.7894737 > \lambda_1a^2 > 5.7817138 \\
n = \frac{3}{2} : & \quad 5.7843552 > \lambda_1a^2 > 5.7829732 \\
n = 2 : & \quad 5.7834061 > \lambda_1a^2 > 5.7831499 \\
n = \frac{5}{2} : & \quad 5.7832276 > \lambda_1a^2 > 5.7831794 \\
n = 3 : & \quad 5.7831939 > \lambda_1a^2 > 5.7831847 \\
n = \frac{7}{2} : & \quad 5.7831875 > \lambda_1a^2 > 5.7831857
\end{align*}
\]

We also give, analogously to (3.51a)–(3.51f), the consecutive differences and their ratios:
\( A^{(1)}a^2 - A^{(1/2)}a^2 = 0.0129878 \) : 10.3 , (3.53a)

\( A^{(3/2)}a^2 - A^{(1)}a^2 = 0.0012594 \) : 7.13 , (3.53b)

\( A^{(2)}a^2 - A^{(3/2)}a^2 = 0.0001767 \) : 5.99 , (3.53c)

\( A^{(5/2)}a^2 - A^{(2)}a^2 = 0.0000295 \) : 5.55 , (3.53d)

\( A^{(3)}a^2 - A^{(5/2)}a^2 = 0.0000053 \) : 5.38 . (3.53e)

\( A^{(7/2)}a^2 - A^{(3)}a^2 = 0.00000099 \)

All is as expected: With each additional iteration the upper bound decreases and the lower bound increases. Notice also in (3.52a)–(3.52g) that the net increase of the lower bound, 0.01446, is less than half of the net decrease in the upper bound, 0.03499. This means that we have made not too bad a choice of \( \lambda_2 \). A related and more striking observation is the contrast between the smooth convergence of the upper limit ratios in (3.51a)–(3.51f), and the initially more rapid descent of the lower limit ratios in (3.53a)–(3.53e); this shows that the second mode is more suppressed in the lower limit, becoming dominant only after several iterations have been performed.

And now the stage is set for an internal determination of \( \lambda_2 \). According to (25.113), the asymptotic value for the ratios displayed in (3.51a)–(3.51f) is \( \lambda_2/\lambda_1 \). The evident convergence makes it plain that \( \lambda_2/\lambda_1 \leq 5.277 \), or, using the five significant figures already established for \( \lambda_1 \) in (3.52g), that \( \lambda_2 a^2 \leq 30.52 \).

First we test whether, as claimed in connection with (25.111), the use of a \( \lambda_2 \) value greater that \( \lambda_2 \) will be made apparent by a qualitative change in the iteration process. Displayed below are the results of lower limit computations employing \( \lambda_2 a^2 = 30.52 \), along with the consecutive differences and their ratios:

\[
\begin{align*}
n = 1/2 & : 5.7753570 \quad : 0.0073961 \\
n = 1 & : 5.7827530 \quad : 18.3 , \\
n = 3/2 & : 5.7831582 \quad : 15.59 , \\
n = 2 & : 5.7831842 \quad : 15.65 , \quad (3.54) \\
\end{align*}
\]
An Example

\[
n = \frac{5}{2} : 5.78318588 : 19.83, \quad : 0.000000084
\]
\[
n = 3 : 5.78318597 : -69.72, \quad : -0.000000012
\]
\[
n = \frac{7}{2} : 5.783185965
\]

The contrast with the ratios in (3.53a)–(3.53e) is eloquent. Clearly the anticipated has happened: After an initial iterative increase in the “lower limits,” that rise has ceased and the convergence to \( \lambda_1 \) from above has begun. As a result we learn that \( \lambda_2 a^2 < 30.52 \), and that

\[
\lambda_1 a^2 < 5.783185965, \quad (3.55)
\]

A computer search program would be most effective in the last step, the quest for the transition from the qualitative behavior of (3.54) to that of (3.53a)–(3.53e), which identifies \( \lambda_2 \). Instead, we present just one example, where the number 30.52 is reduced about 0.2\% to 30.45:

\[
n = \frac{1}{2} : 5.7752353 : 0.0074987
\]
\[
n = 1 : 5.7827340 : 17.8, \quad : 0.0004209
\]
\[
n = \frac{3}{2} : 5.7831548 : 14.6, \quad : 0.0000288
\]
\[
n = 2 : 5.7831836 : 13.2, \quad (3.56)
\]
\[
n = \frac{5}{2} : 5.7831858 : 12.1, \quad : 0.00000018
\]
\[
n = 3 : 5.78318594 : 10.6 \quad : 0.00000017
\]
\[
n = \frac{7}{2} : 5.783185961
\]

The situation has become normal, and so we know that \( \lambda_2 a^2 > 30.45 \), and that

\[
\lambda_1 a^2 > 5.783185961. \quad (3.57)
\]

What have we accomplished? The best determination of \( \lambda_1 a^2 \) in (3.52g) can be presented as

\[
\lambda_1 a^2 = 5.7831866 \pm 0.0000009, \quad (3.58)
\]
an accuracy of about one part in a ten million. Now, without any additional input, $\lambda_1 a^2$ has, according to (3.55) and (3.57), been located at

$$\lambda_1 a^2 = 5.783185963 \pm 0.000000002,$$  \hspace{1cm} (3.59)

an accuracy of 3 parts in 10 billion.

We have refrained from explicit use of the true values of $\lambda_1$ and $\lambda_2$, which are the squares of (CE19.127) and (CE19.129), respectively:

$$\lambda_1 a^2 = 5.78318596297,$$  \hspace{1cm} (3.60a)

$$\lambda_2 a^2 = 30.4712623438.$$  \hspace{1cm} (3.60b)

That our choice of $\lambda_2$ in (3.56) is very close to $\lambda_2$ is quite apparent in the significantly increased rate of convergence there, as compared with (3.53a)–(3.53c). What happens if we use the actual $\lambda_2$ value? The result of $\lambda^{(7/2)}$ increases almost indiscernibly, the successive ratios somewhat more – in particular, the final ratio, 10.6, is raised to 13.2. And that is reasonable, for then these ratios are converging toward [(CE25.125)]

$$\frac{\lambda_3}{\lambda_1} = 12.949092.$$  \hspace{1cm} (3.61)