Physics 5583. Electrodynamics II.
First Examination
Spring 2004

February 20, 2004

Instructions: This examination consists of three problems. If you get stuck on one part, assume a result and proceed onward. Show all your work. Do not hesitate to ask questions. GOOD LUCK!

1. A point charge $e$ is placed at the center of a spherical cavity of radius $a$ embedded in a uniform dielectric. That is, with origin chosen at the center of the cavity, the permittivity satisfies

$$
\epsilon(r) = \begin{cases} 
1, & r < a, \\
\varepsilon, & r > a,
\end{cases}
$$

with $\varepsilon$ a constant. Suppose that besides the point charge at the center of the sphere, there is otherwise no free charge.

(a) Calculate the electric field $\mathbf{E}$ at each point in space.

(b) Show that the pressure exerted by the charge on the walls of the cavity, the radial integral of the radial component of the force density

$$
f = -\frac{E^2}{8\pi} \nabla \epsilon,
$$

is

$$
P = -\frac{\varepsilon - 1}{8\pi} \left( E^2_{\perp} + \frac{1}{\varepsilon} E^2_{r} \right)_{r=a^-}. $$
[Hint: remember the boundary conditions satisfied across a dielectric interface by the electric field $E$ and the electric displacement $D$.]

(c) Insert the value of the electric field found in Problem 1a to find the total stress on the spherical wall of the cavity, $S$,

$$\mathcal{F} = \int_S dS P.$$

(d) Compute the radial-radial component of the stress tensor $T_{rr}$, and compute the pressure on the surface from

$$T_{rr}\bigg|_{r=a-} - T_{rr}\bigg|_{r=a+}.$$

Does the result agree with that found in Problem 1c?

(e) Show in general (arbitrary $E$) that the discontinuity of the stress tensor found in Problem 1d coincides with the formula for the pressure found in Problem 1b.

2. Consider a perfectly conducting spherical shell of radius $a$. As shown in class, the Green’s function inside the shell is

$$r, r' < a : \quad G(r, r') = \frac{1}{|r - r'|} - \frac{a}{r'} \frac{1}{|r - r'|},$$

where if, in polar coordinates, $r' = (r', \theta', \phi')$, the image point is located at $r' = (a^2/r', \theta', \phi')$. The first term in the Green’s function is the Green’s function in vacuum (Coulomb potential),

$$G_0(r, r') = \frac{1}{|r - r'|}.$$

(a) Argue that since the self-energy of the charge is not observable, the energy of interaction of a charge distribution $\rho(r)$ with a material background (in this case, the shell) is given by

$$E_{\text{int}} = \frac{1}{2} \int (dr)(dr')\rho(r)[G(r, r') - G_0(r, r')]\rho(r').$$

(b) Calculate $E_{\text{int}}$ explicitly for a point charge $e$ located at $r_0$, $r_0 < a$. 

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(c) Compare the result of the previous part with the energy of interaction between the point charge and the image charge.

(d) From the energy calculate, by taking an appropriate derivative, the force exerted on the point charge by the sphere. How does this compare with the force exerted by the fictitious image charge on the point charge?

(e) If \( r_0 = 0 \), show that the energy computed in Problem 2b is consistent with the stress computed in Problem 1c, if the limit \( \varepsilon \to \infty \) there is taken, corresponding to a perfectly conducting boundary.

3. This problem gives an alternate derivation of the addition theorem for spherical harmonics.

(a) Argue that because \( r^l P_l(\cos \gamma) \) is a solution of Laplace’s equation, and so is \( r'^l P_l(\cos \gamma) \) [that is, \( \nabla^2 r^l P_l(\cos \gamma) = 0 \)], we must have

\[
P_l(\cos \gamma) = \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi')
\]

in terms of \( 2l + 1 \) real coefficients \( a_{lm} \). Here, \( \gamma \) is the angle between the directions specified by \( (\theta, \phi) \) and \( (\theta', \phi') \),

\[
\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'),
\]

so that \( \cos \gamma \) depends only on the difference between \( \phi \) and \( \phi' \).

(b) Now set \( \theta' = \theta, \phi' = \phi \), so that \( \gamma = 0 \), and deduce by integrating over all angles \( \theta \) and \( \phi \) that

\[
4\pi = \sum_{m=-l}^{l} a_{lm}.
\]

(c) Then multiply the expansion of \( P_l(\cos \gamma) \) given in part 3a by its complex conjugate, and integrate over \( \theta, \phi, \) and \( \theta', \phi' \), separately, to deduce that

\[
\frac{(4\pi)^2}{2l + 1} = \sum_{m=-l}^{l} a_{lm}^2.
\]
(d) Combine the results of parts 3b and 3c to conclude that

\[ a_{lm} = \frac{4\pi}{2l + 1}, \]

which implies the addition theorem. [Hint: Think about fluctuations from the mean.]