1. Consider a waveguide with perfectly conducting boundaries which are disconnected, so that it can support T modes. This is a mode which has neither a longitudinal magnetic nor a longitudinal electric field,

\[ H_z = E_z = 0, \]

and is both an E and an H mode,

\[ \mathbf{E} = \nabla \perp \phi = \nabla \times e \psi, \quad (1) \]

where \( e \) is a unit vector along the axis of the guide, the \( z \) axis.

(a) Show that if this is true, the eigenvalues of the mode functions are zero:

\[ \nabla^2 \perp \phi = \nabla^2 \perp \psi = 0. \quad (2) \]

(b) Show from the conditions satisfied by \( \mathbf{E} \) on the boundary that the circumferential derivative of \( \phi \) vanishes,

\[ \frac{\partial \phi}{\partial s} = 0, \]

so that \( \phi \) is constant on each bounding curve \( C \), and that the normal derivative of \( \psi \) vanishes:

\[ \frac{\partial \psi}{\partial n} = 0. \]
(c) Now show in Cartesian coordinates that Eq. (1) implies

\[ \frac{\partial}{\partial x} \phi = \frac{\partial}{\partial y} \psi, \quad \frac{\partial}{\partial y} \phi = -\frac{\partial}{\partial x} \psi, \]

and then show that \((\xi = x + iy)\)

\[ F(\xi) = \phi(x, y) + i\psi(x, y) \]

is an analytic function of \(\xi\). (That is, it has a complex derivative, \(\frac{dF}{d\xi}\).)

(d) Now consider parallel plates situated at \(x = 0\) and \(x = a\). Show that a way of satisfying the boundary conditions found in part 1b is to take

\[ \phi = \phi(x), \quad \psi = \psi(y). \]

From Eq. (2) determine the form of the functions \(\phi(x)\) and \(\psi(y)\), and verify that the assertion given in part 1c is valid.

(e) Consider a coaxial cable consisting of a cross section having an inner circular conductor of radius \(a\) and an outer concentric circular conductor of radius \(b\), \(b > a\). Show that the boundary conditions in part 1b are satisfied if in cylindrical coordinates \((\rho, \theta)\)

\[ \phi = \phi(\rho), \quad \psi = \psi(\theta), \]

and determine the form of \(\phi(\rho)\) and \(\psi(\theta)\) from Eq. (2). Again, how is the analytic form in part 1c achieved?

2. Consider a cylindrical waveguide with a cross section in the shape of a square of side \(a\).

(a) Using a convenient coordinate system, write down the mode function \(\phi\) and \(\psi\) for the E and H modes.

(b) What are the corresponding eigenvalues?

(c) What is the longest intrinsic wavelength

\[ \lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \]

that can propagate in the guide as an E mode?
(d) What is the longest wavelength that can propagate in the guide as an H mode?

(e) Which of these cutoff wavelengths is the longest?

(f) Consider the lowest H mode, which has the following form for the electric and magnetic fields: \( \eta = \sqrt{\epsilon/\mu}, \ c = 1/\sqrt{\epsilon\mu} \)

\[
H_z = i\frac{\eta}{k} \gamma^2 \psi(x, y)V(z), \\
H_\perp = -\nabla_\perp \psi(x, y)I(z), \\
E_\perp = e \times \nabla \psi(x, y)V(z).
\]

Write down \( \psi \) from part 2a, but so normalized that

\[
\int_0^a dx \int_0^a dy \nabla_\perp \psi \cdot \nabla_\perp \psi = 1.
\]

(g) From the explicit form for the electric field so found, determine the flux of energy from the Poynting vector

\[
S = \frac{1}{2} \text{Re} E \times H^*.
\]

(Why is the factor of 2 present?) Determine the flux of energy

\[
\int dx \ dy \ S_z
\]

in terms of the functions \( I(z), V(z) \).

Problems in *Classical Electrodynamics*, Chapter 44: 2, 3, 6, 7, 8, 9, 10, 11, 12.