Physics 5013. Homework 8
Due Wednesday, December 13, 2006

November 21, 2006

1. Suppose we have a second-order differential operator of the form

\[ L = \frac{1}{f} \frac{d}{dx} \left( f \frac{d}{dx} \right) + q, \]

where \( f \) and \( q \) are functions of \( x \). If \( y_1 \) and \( y_2 \) are independent solutions of

\[ Ly = 0, \]

the Wronskian

\[ \Delta(y_1, y_2) = y_1 y_2' - y_2 y_1' \]

is different from zero. Prove that

\[ \frac{d}{dx} \Delta = -\Delta \frac{d}{dx} \ln f, \]

and that

\[ y_2(x) = \Delta(x_0) f(x_0) y_1(x) \int_{x_0}^{x} \frac{du}{f(u) y_1^2(u)}, \]

where \( x_0 \) is a point at which

\[ y_2(x_0) = 0, \quad y_1(x_0) \neq 0, \]
\[ f(x_0) \neq 0, \quad y_1'(x_0) \neq 0. \]

2. Recall that the Bessel functions of integer order are defined by

\[ e^{(x/2)(z-1/z)} = \sum_{m=-\infty}^{\infty} z^m J_m(x), \]
or, with \( x = kr, \ z = ie^{i\phi} \).

\[
e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} \imath^m e^{im\phi} J_m(kr).
\]

Use this expression in the two-dimensional completeness statement for the functions

\[
\frac{1}{2\pi} e^{ik \cdot r},
\]

that is,

\[
\int \frac{(dk)}{(2\pi)^2} e^{ik \cdot r} e^{-ik \cdot r'} = \delta(r - r'),
\]

where the right-hand side is a two-dimensional delta function, which in polar coordinates is

\[
\delta(r - r') = \frac{1}{r} \delta(r - r') \delta(\theta - \theta'),
\]

and \((dk)\) is the two-dimensional integration element, which is correspondingly given in polar coordinates as

\[
(dk) = k \, dk \, d\alpha.
\]

In this way derive the completeness property of the Bessel functions,

\[
\int_0^{\infty} k \, dk \, J_m(kr) J_m(kr') = \frac{1}{r} \delta(r - r').
\]

3. Determine, directly, the one-dimensional Green’s function \( G_k(r, r') \) for the Bessel differential operator of order zero; that is, solve

\[
\frac{d}{dr} \left( r \frac{dG_k}{dr} \right) + k^2 r G_k = \delta(r - r'), \quad 0 \leq r \leq a,
\]

subject to the boundary condition

\[
G_k(a, r') = 0.
\]

Show that \( G_k \) is singular wherever \( k = k_n \), where

\[
J_0(k_n a) = 0.
\]
From the behavior of $G_k$ at this singularity determine the normalization integral

$$\int_0^a r J_0^2(k r) \, dr.$$  

[Hint: It is necessary to use both the regular solution to Bessel’s equation of order zero, $J_0$, and the irregular solution, $N_0$. The result of problem 1, as well as the asymptotic behaviors

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{4} \right), \quad x \gg 1,$$

$$N_0(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\pi}{4} \right), \quad x \gg 1,$$

will be helpful.]

4. Find the Green’s function for the two-dimensional Helmholtz equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right) G(x, y; x', y') = \delta(x - x')\delta(y - y')$$

in the interior of a square of side $a$, expressed as an eigenfunction expansion. With the origin of coordinates chosen to be one corner of the square, the boundary conditions are as follows (see Fig. 1):

$$G(0, y; x', y') = 0,$$
$$G(a, y; x', y') = 0,$$
$$\frac{\partial}{\partial y} G(x, 0; x', y') = 0,$$
$$\frac{\partial}{\partial y} G(x, a; x', y') = 0.$$  

5. (a) Find the Green’s function for Laplace’s equation,

$$\nabla^2 G(r, r') = \delta(r - r'),$$

in a three-dimensional region lying between the two planes $x = 0$ and $x = a$ as shown in Fig. 2 with the boundary conditions
Figure 1: Boundary conditions for the Green’s function in Problem 4.

Figure 2: Coordinate system for Problem 5.
\[ G(0, y, z; x', y', z') = 0, \]
\[ G(a, y, z; x', y', z') = 0, \]

using the following method: Show that \( G \) can be written in the form

\[ G(r, r') = \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z e^{ik_y(y-y')} e^{ik_z(z-z')} g_{k_\perp}^2 (x, x'), \]

where \( k_\perp^2 = k_y^2 + k_z^2 \), and find \( g_{k_\perp}^2 \) in closed form by using the discontinuity method.

(b) By examining the singularities of \( g_{k_\perp}^2 (x, x') \) with respect to \( k_\perp^2 \) find the normalized eigenfunctions and eigenvalues of \( \frac{d^2}{dx^2} \) subject to the homogeneous Dirichlet boundary conditions at \( x = 0 \) and \( x = a \).

(c) Using the result of (b) and the generating function for the Bessel functions

\[ e^{\frac{x}{2}(z+\frac{1}{z})} = \sum_{m=-\infty}^\infty z^m J_m(x), \]

prove that

\[ G(r, r') = -\frac{1}{2\pi} \int_0^{\infty} dk k J_0(kR) \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a}}{k^2 + \left( \frac{n\pi}{z} \right)^2}, \]

where \( R^2 = (y - y')^2 + (z - z')^2 \).