Chapter 8 Lecture Notes
Physics 2414 - Strauss

Formulas:
$v = \Delta l / \Delta t = r\Delta\theta / \Delta t = r\omega$
$a_T = \Delta v / \Delta t = r\Delta\omega / \Delta t = r\alpha$
$a_C = v^2 / r = \omega^2 r$
$\omega = \omega_0 + \alpha t$
$\theta = \omega_0 t + (1/2)\alpha t^2$
$\theta = (1/2)(\omega_0 + \omega)t$
$\omega^2 = \omega_0^2 + 2\alpha\theta$
$\tau = rF \sin\theta$
$I = \Sigma mr^2$
$I = Mk^2$
$\mathbf{L} = I\omega$
$\Sigma\tau = I\alpha = \Delta\mathbf{L}/\Delta t$
$K = (1/2)I\omega^2$

Main Ideas:
1. Angular Quantities
2. Rotational Kinematics
3. Torque and Rotational Inertial
4. Rotational Kinetic Energy
5. Angular Momentum

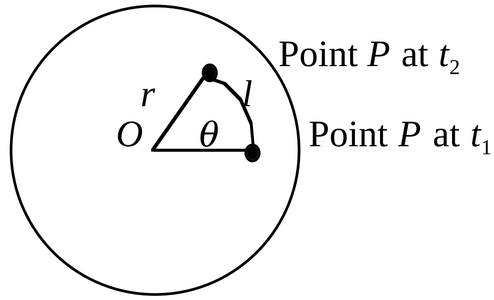
1. Angular Quantities

This chapter will again deal with rotation, like chapter 5 did. We will look rigid bodies that have purely rotational motion. Rigid bodies are objects that have a definite shape. Purely rotational motion means that all points in the object rotate in circles. They all rotate around a fixed **axis of rotation**. Finally, we will combine rotational motion with translational motion. The concepts in this chapter are very similar to the concepts we learned in chapter 2 through 7 regarding kinematic motion in one and two dimensions, Newton's Laws, the conservation of mechanical energy, and the conservation of momentum. As we develop concepts in this chapter, you should try to relate the concepts to those in previous chapters. It will help you understand this chapter.

1.1 ANGULAR DISPLACEMENT

When an object moves along a straight path, we describe how far it has moved by its displacement. When an object rotates we describe how far it has rotated by its angular displacement θ.

The mathematics of circular motion is much simpler if we measure the angle in radians rather than degrees. One radian as defined as an angle whose arc length is equal to its radius, or in general



$\theta = l/r$

where r is the radius of the circle and l is the arc length subtended by the angle θ. If the angle does a complete revolution, then $l = 2\pi r$, which is the circumference. So 360° = 2π radians, and 1 radian is about 57.3°. **Note that as the object rotates, every point on the object undergoes the exact same angular displacement.** The angular displacement θ, will play the same role in angular kinematics that the displacement x, played in linear kinematics. It is customary to set counterclockwise rotations to have a positive angular displacement and clockwise rotations to have a negative angular displacement.

1.2 Angular Velocity

The angular velocity is defined in relationship to the angular displacement in the same way that the linear velocity was defined in relationship to the linear displacement. The average angular velocity is given by the Greek letter omega (ω), and is defined as the rate of change of the angular displacement.

$\omega = \Delta\theta / \Delta t$

The instantaneous angular velocity is given by

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

1.3 Angular Acceleration

Likewise, the average angular acceleration is defined as the rate of change of the angular velocity, and is given by the Greek letter alpha (α).

$\overline{\alpha} = \Delta\omega / \Delta t$

and the instantaneous angular acceleration is given by

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

1.4 Relationship Between Angular and Linear Quantities

Before using these angular quantities let's look at the relationship between them and the linear quantities we have used. We know that velocity is defined as the change in distance divided by the change in time. (In this case, the velocity is in the direction tangent to the circumference of the circle: the tangential velocity, v_T, so the distance is given by the arc length, l).

$$v_T = \Delta l / \Delta t = r\Delta\theta / \Delta t = r\omega$$

Also, the tangential component of the acceleration is

$$a_T = \Delta v_T / \Delta t = r\Delta\omega / \Delta t = r\alpha$$

Recall the definition of centripetal acceleration.

$$a_C = v_T^2/r = \omega^2 r$$

with total acceleration: $\mathbf{a} = \mathbf{a}_T + \mathbf{a}_C$

and since $\mathbf{a}_T$ and $\mathbf{a}_C$ are at right angles to each other, the magnitude of the total acceleration is given by,

$$a = \sqrt{a_T^2 + a_C^2}$$

1.5 OTHER USEFUL DEFINITIONS

Frequency is defined as the number of revolutions per unit time. To change from frequency to angular frequency, I must multiply frequency by 2π since one revolution is 2π radians. (1 rev/s)*(2π radians/rev) gives radians/sec which is the units of angular velocity.

$$\omega = 2\pi f$$

The **period** is the time required to make one complete revolution, so

$$T = 1/f.$$

Problem: How fast is the outer edge of a CD (at 6.0 cm) moving when it is rotating at its top speed of 22.0 rad/s?

2. Rotational Kinematics

The definitions of the angular quantities are analogous to the definitions of linear quantities with θ playing the role of x, ω playing the role of v, and α playing the role of a. Consequently, the kinematic equations derived in chapter 2 are valid for rotational motion with constant angular acceleration using the same derivations done in chapter 2. We get,

$$\omega = \omega_0 + \alpha t$$

$\omega^2 = \omega_0^{\,2} + 2\alpha\theta$
$\theta = \omega_0 t + (1/2)\alpha t^2$
$\theta = (1/2)(\omega_0 + \omega)t$

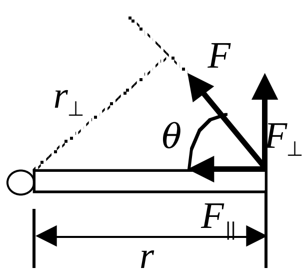
Problem: A compact disk player starts from rest and accelerates to its final velocity of 3.50 rev/s in 1.50 s. What is the disk's average angular acceleration?

Problem: How many rotations does the CD from the previous problem make while coming up to speed?

3. Torque and Rotational Inertia

We have talked about the description of angular motion. Now we talk about the causes or dynamics of angular motion. What causes something to rotate? For translational motion, we found that a force caused an acceleration. That is a force caused an object to move. For rotational motion, it also takes a force to start something rotating. However, there is more to it than that. The location that I apply the force is important. If I want to start a wheel rotating, I can not apply the force at the axle of the wheel. It will not rotate. To start something rotating, I must apply a force that is some distance from the axis of rotation. The perpendicular distance from the axis of rotation is called the **lever arm.** The torque is defined as the force times the lever arm.

$$\tau = r_\perp F$$
$$= rF\sin\theta$$
$$= rF_\perp$$

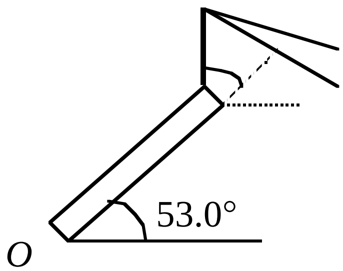


In general we write torque as
$\tau = rF\sin\theta$

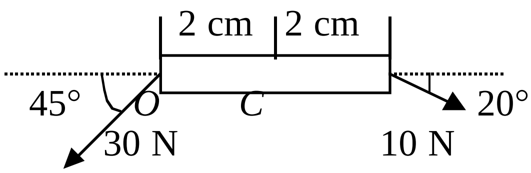
where θ is the angle between the direction of the force and a line drawn from the axis of rotation to the force. If I were to push along the direction of $F_{\parallel}$ then there would be no rotation around the hinge, since the force is directed right through the axis of rotation. If I were to push along $F_\perp$ then there would be a rotation. Consequently, the only component of the force F which causes a rotation is the component perpendicular to the lever arm (r), which is given by F

sinθ. So what produces a greater torque? A force applied at some distance from the axis of rotation at some angle other than directly through the axis of rotation. If I increase the lever arm I get a greater torque. I need to do this to take off the drain plug from my car. If I increase the force, the torque will increase. I can also maximize the torque by making the angle θ = 90°. Because torque has a distance in its definition (the lever arm), the torque is only defined around a certain axis of rotation. The torque around two different axes of rotation may be very different.

<u>Problem:</u> A crane picks up a heavy steel beam that is 10.0 m long. The tensions in the cable is 2.00×10^4 N. What is the torque around point *O*?



<u>Problem:</u> Calculate the torque around an axis perpendicular to the paper through (a) point *O* , and (b) point *C*.



Recall in our study of linear motion that we said that the property of a body that resisted a change in velocity was called mass. Even if there was no gravity, the mass of a body would resist a change in velocity. That is what Newton's second law says $\Sigma\mathbf{F} = m\mathbf{a}$. There is also a property of a body which resists a change in angular velocity. It is called the **moment of inertia**. If a body has a large moment of inertia, then it is difficult to change its angular velocity. If it has a small moment of inertia, it is easier to change its angular velocity. The moment of inertia for any object depends on a number of factors including the object's mass, its shape, and the axis of rotation. Let's first calculate the moment of inertia for a simple object like a sphere at the end of the string. Since the sphere has much more mass than the string, we will neglect the mass of the string. We look at the tangential forces and accelerations.

$F_T = ma_T$
$F_T = mr\alpha$

$rF_T = mr^2\alpha$
$\tau = mr^2\alpha$

In this case, the quantity mr^2 is called the moment of inertia and given the symbol I. So this becomes $\tau = I\alpha$. If we have more then one source of torque, we get

$$\Sigma\tau = I\alpha$$

which is the rotational equivalent of Newton's second law. The reason we write I rather than mr^2 is because I depends on different factors and although it is mr^2 for a sphere on a string, it is not mr^2 for every object. In general, we can write the moment of inertia for a collection of particles as

$$I = \Sigma mr^2$$

This can be used to calculate the moment of inertia of a baton, for instance, but not for many other extended objects. In that case, calculus is needed. Page 204 lists the moment of inertia for a few shapes. In any problem you have to do, you will be given the equation for the moment of inertia.

<u>Problem:</u> Suppose a baton is 1.0 m long with each end weighing 0.3 kg. Neglect the mass of the bar. What is the moment of inertia for a baton (a) spinning around its center? (b) spinning around one end of the baton?

Note that the moment of inertia depends on where the axis of rotation is.

Another quantity is the **radius of gyration.** It is given the symbol k, and is defined as the location on the object that would have the same moment of inertia as the original object if all the mass where located at k. The relationship between the moment of inertia and the radius of gyration is always given by

$$I = Mk^2.$$

Let's now solve some problems using Newton's second law in this form for rotational motion. We use it the same way as we used Newton's second law for purely translational motion. We must draw a free body diagram, determine the torques and the accelerations, and solve for the unknown quantities.

Up until now we have not been concerned about *where* a force acted on an object. But because torques produce rotation around some axis that depends on the distance from where the force is applied, it is important to look at where a force is applied on an object. For the force of gravity, we assume the force is applied at the center of gravity of an object. Other forces, like tension, friction, and normal forces are applied where the objects contact.

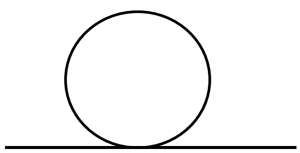
<u>Problem:</u> A cylindrical 3.00 kg pulley with a radius of R=0.400 m is used to lower a 2.00 kg bucket into a well. The bucket starts from rest and falls for 3.00 s. (a) What is the linear acceleration of the falling bucket? (b) How far does it drop? (c) What is the angular acceleration of the cylinder?

So these problems are solved just like other problems using Newton's second law. Draw force diagrams. Set $\Sigma F = ma$ and $\Sigma\tau = I\alpha$. Break vectors into components if you need to. Solve for the unknown quantities. Remember that translational motions are produced by forces and rotational motions are produced by torques.

Sometime objects are constrained to rotate around a certain axis because they are attached by hinges or something similar. Sometimes objects are free to rotate about any axis because the have no constraints, like when you throw an object in the air. An object free to rotate about any axis will always rotate around its center of mass.

4. Rolling Motion

When an object is rolling we can observe some aspects about its motion.



Suppose the object is moving to the right with a speed of v. Look at the linear speed of various points on the object. Where the object touches the ground its linear speed is zero. At the center of the wheel the linear speed is v and at the top of the wheel the velocity is $2v$. The angular velocity, then, is given by

$$v = \omega r$$

where v is the linear speed of the wheel. What causes the wheel to rotate? It is the force of static friction on the bottom. It is static friction because the place where the wheel touches does not move sideways with respect to the ground. The static friction produces a torque around the middle point of the wheel. That is the axis of rotation.

5. Rotational Kinetic Energy

We have learned that when something is moving it has a kinetic energy which is equal to $(1/2)mv^2$. When something is rotating it has a kinetic energy which is equal to (1/2)$I\omega^2$. If it has both translational and rotational motion, then it has both forms of kinetic energy, as well.

Problem: Two bicycles roll down a hill which is 20 m high. Both bicycles have a total mass of 12 kg and 700 mm diameter wheels (r=.350 m). The first bicycle has wheels that weigh .6 kg each, and the second bicycle has wheels that weigh .3 kg each. Neglecting air resistance, which bicycle has the faster speed at the bottom of the hill? (Consider the wheels to be thin hoops).

The only friction is static friction, so there are no non-conservative forces. (Static friction involves no motion and since work is defined as $W = Fd$, when there is no distance involved, there is no work/energy used).

Why does the bike with the lighter wheels have the faster speed? Because each bike starts and ends with the same potential energy so they also must have the same total kinetic energy at the end. More of the potential energy goes into rotational kinetic energy and less into translational kinetic energy when the wheel has more mass. Because the translational kinetic energy is less, the speed is smaller.

6. Angular Momentum

In the same way that linear momentum is defined as $p = mv$, so the angular momentum of a rigid body rotating is defined as $L = I\omega$. Similarly, Newton's law for rotational motion can be written as

$$\Sigma\tau = I\alpha = \Delta L/\Delta t$$

and just as linear momentum is conserved if there is no net external force, so angular momentum is conserved (it does not change) if there is no net external torque. The law of the conservation of angular moment says:

The total angular momentum of a rotating body remains constant if the net external torque acting on it is zero.

Some examples of this include an ice skater who is rotating and pulls her arms in or a diver who goes into a tucked position. Why do they rotate faster?

$$\Sigma L_i = \Sigma L_f$$
$$\Sigma I\omega_i = \Sigma I\omega_f$$
$$\Sigma m r_i^2 \omega_i = \Sigma m r_f^2 \omega_f$$

So as the distance from the axis of rotation decreases, the angular velocity (ω) must increase for L to be the same.

Problem: A student is sitting on a swivel seat and is holding a 2.0 kg weight in each hand. If he is rotating at 1 rev/s (6.28 rad/s) when the weights are held in outstretched arms .75 m from the axis of rotation, how fast is he rotating when he pulls the weights in to the axis of rotation? (The rest of his body can be approximated as a cylinder with mass of 72 kg and radius of .25 m).

7. Vector Nature of Angular Quantities

The angular equivalent of all of the linear quantities which are vectors are also vectors. This includes all of the angular quantities we have discussed except moment of inertia. The direction is chosen arbitrarily by the right hand rule. Your fingers rotate in the direction of the rotational motion and your thumb points in the direction of the vector. Note, however, that the relationship between angular and linear quantities (like $v = r\omega$, $a_T = r\alpha$, and $a_C = \omega^2 r$) are not vector equations. The linear quantities point in the direction of motion which changes as the object rotates, but the angular quantities point in the direction given by the right hand rule.

8. Inertial and Non Inertial Frames: The Coriolis “Force”

Any frame of reference where Newton’s laws appear to work is called an inertial frame of reference. Sometimes it may appear that his laws do not work. For instance, in a rotor carnival ride, we may think that we are being thrown outward. However, there is no force which is throwing us outward, instead we are being accelerated inward by the centripetal acceleration provided by the wall of the ride. This reference frame, where we think there is a force, but there really is not one is called a noninertial reference frame, and the force we think we feel is called a psuedo-force. The Coriolis force is one such force. Suppose I have a plate rotating around its center. Points on the inside of a plate have a lower tangential velocity than those on the outside of the plate. Consequently, something thrown outward appears to curve in the opposite direction of rotation. This is an important psuedo-force in understanding weather and in understanding the path of ballistic shells. Ships at sea used to carry tables of values that they could use to estimate the Coriolis force depending on latitude and direction they wanted the shell to travel. Then they could correct for the Coriolis force.