Statistics and Distributions

1 Purpose:

This lab is designed to acquaint you with the concepts of statistical distributions and partition functions. It is divided into two parts.

- Lab Work: The part done in lab requires you to work in cooperative groups of four. (No group should have more than five members). You will be asked a series of questions (below), and be asked to make some observations. These questions and observations must be handled as a group; everyone should eventually write down the same answers (though wording may vary). The discussion itself is an important part of this lab. You must try to obtain some group consensus before moving on.
- **Question 1:** Assume that you have a collection of 64 coins, and randomly distribute them on a chess board. Describe in detail what you think is the most probable spatial distribution.

2 Description of Activity:

One person from the group should obtain a set of dice from the instructor. You need n "n-sided" dice, *i.e.* six six-sided dice, eight eight-sided dice, or ten ten-sided dice. It does not matter which set you choose. If you choose a die with a small number of sides (four or six) you are expected to take many more trials, since they are easier to sum. You will then have one person roll the dice and three others gather different statistics. By dividing up the task you can investigate several things at once.

• One person should keep track of how many times each number shows up on the dice. Use the histogram labeled "frequency of occurrence" and record how many times each

number occurs. You should take data by "x-ing" off a box each time a number appears on the top face of a die. Every twenty throws you should change the color of the pen you are using so that you can see how the distribution changes with the number of throws. We will call this distribution of the number of times a number appears on a die P(n).

- One person should keep track of the total of the dice. On each throw of the dice add them up, and mark it on the histogram labeled "sum of dice". Again, change the color of the pen used every ten rolls so that you can see how the distribution takes shape. We will call this distribution S(n).
- One person should keep track of the numbers of singles, pairs, triples etc. of dice in the throw. For example, if you roll six six-sided dice and obtain the set {1, 2, 1, 3, 4, 5}, this is marked as one pair and 4 singles. We will call this function N(n).

Note that the symbol n means different things in the above functions! The function P(2) is the number of times you role "2"; the function N(2) is the number of times you roll pairs!

Question 2: Sketch below **before rolling the dice** what you expect for the distributions P(n), S(n), and N(n). Make sure everyone in the group agrees on the answer before proceeding.

Record your rolls in histogram form on the sheets provided, shading in boxes for each occurrence. When recording your rolls in histogram form, switch colors after every 20 rolls. This will allow you to see how the distribution takes shape. You should try to gather at least 60 rolls.

3 Outcome of experiment:

Answer these questions only after taking the data. Again, make sure that you all understand and

agree on the answers before moving on.

Question 3: Compare your expected graph for P(n) with that obtained by rolling the dice. Describe any differences. Should you expect an absolutely "flat" distribution? If not, how large are the fluctuations? Do they get larger with the number of rolls?

Question 4: Calculate the mean or average number of times a die face appears, \overline{N} , by summing P(n) for all values of *n* and then dividing by the number of faces on your die, *N*. You should also calculate the standard deviation

$$\sigma = \sqrt{\sum_{n=1}^{n=N} \frac{1}{N} (P(n) - \overline{N})^2}$$

Question 5: Compare your expected graph for S(n) with that obtained by rolling the dice. Describe any differences. How sharply peaked is the distribution? Can you estimate how far you must move off the peak number so that the frequency of occurrence falls in half? How does the distribution change as the total number of rolls increases? **Question 6:** Compare your expected graph for N(n) with that obtained by rolling the dice. Describe any differences. How would you describe the dependence of the function N(n)? Is it rapidly varying? Guess some functional form for N(n) as a function of n.

Question 7: Assume your are given a distribution P(N). Can you use it to find $N_{tot} = N_{dice} X$ N_{rolls} , the total number of die rolls? Is it possible to do so exactly, using S(N)? Approximately? **Question 8:** Assume that you are given a distribution N(n). Can you use it to find N_{tot} (defined above)? Write down an expression for N_{tot} using N(n).

4 The Physics Connection:

Assume (in order to pick a concrete case) that you were rolling eight dice that each had eight sides. Now imagine that you had a system of eight particles and eight "packets of energy" each of size u, and that you wanted to distribute the energy randomly among the particles. One way to do that is to use the dice: a particle is given a number of "packets" equal to the number of times its number appears in the dice roll. For example the dice roll {1, 2, 4, 6, 3, 2, 2, 1} would mean that particle 2 has an energy 3u, particle 1 has an energy 2u, particles 3, 4, and 6 have an energy u and particles 5, 7 and 8 have no energy at all. The function N(n) then describes the *distribution* of particle energies. Such a function is called a **partition function**.

Question 9: Using your guess of the functional form for N(n) in Question 5, what is the ratio of $N(n+\Delta n)/N(n)$? What does this say about how many particles have high energy? How would this distribution depend upon particle energy?

The toy system of eight particles is a model of an isolated system, because the energy of the system is fixed we just redistribute it with every roll of the dice. In most real systems we do not have a constant energy but a constant temperature, because the system we are examining is small, and exchanges energy with the rest of the world, but in such a fashion that the temperature is fixed. The distribution of energies you obtained above is related to, but not identical to the distribution found at fixed temperature; this distribution is the **Boltzmann distribution**.

Question 10: Look at the fluctuations in P(n) for 20, 40 etc. rolls. (You don't need to calculate the standard deviation for each set, just get an intuition for what is happening.) Can you estimate how the fluctuations vary as you increase the number of rolls? How about the *relative size* of the fluctuations, σ/P_{avg} , where P_{avg} is the average value of P(n) for that number of rolls?

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