Heat Engine Problem

Your group has been hired as consultants by Oklahoma Gas and Electric to help design their power plant. Their turbines use a monatomic gas and follow the cycle below:

A→B: This is an isotherm at a high temperature T_H. Exactly how high will be up to you to determine. The system expands from a volume V_A to V_B. Design constraints force $V_B = 4V_A$.

B→C: This is an isochore on which the system is cooled to room temperature, $T_C = 300$ K.

C→D: This is an isotherm at the low temperature T_C. The system is returned to its original volume V_A.

D→A: This is an isochore on which the system is returned to its original temperature, T_H.

You are hired to answer some or all of the following questions:

1. Draw this cycle.

2. What is the efficiency of the system? First write your answer as a function of T_H, V_A, V_B, and T_C. Then put in the correct values for V_A, V_B, and T_C so your answer is a function of T_H.

3. How does this compare with the efficiency of a Carnot engine acting between T_H and T_C? Is it a big difference or a little difference?

4. Plot (1) the efficiency of this engine, (2) the Carnot efficiency of an engine acting between the temperature extremes, and (3) the ratio of $\varepsilon/\varepsilon_C$ as a function of T_H, all on the same graph.

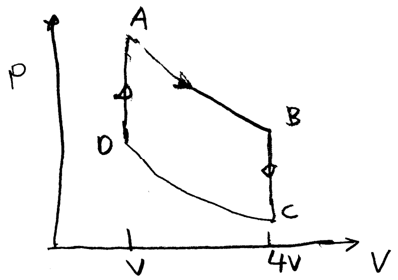
5. Estimate the high temperature needed for this cycle to reaches 80% of its maximum efficiency (not the Carnot efficiency)?

6. Estimate the high temperature that this cycle reaches 80% of the Carnot efficiency at that same high temperature?

7. OG&E can bill for energy at a rate of 10^{-3} dollars/Joule. The cost of energy to run the engine depends on its operating temperature. The hotter it runs, the more expensive it is to operate. The cost of energy is $(T_H)^4 \times 10^{-7}$ dollars/(Joule·K). Calculate the ratio of money earned/money spent as a function of T_H. At approximately what T_H is it a maximum? (Again, drawing a plot may be helpful. Note that the energy OG&E needs to pay for is the energy to run the engine, but they can only charge based on the work output of the engine.)

8. If they insulate the turbines, they can redesign them to act more like a Carnot engine. How would the ratio of money earned/money spent change? A little or a lot?

1)



$Q = \Delta U + W$ $\qquad T_c = 300K$

$T_A = T_B = T_H$

$T_C = T_D = T_c$

$V_D = V_A \;,\quad V_C = V_B$ $\qquad V_B/V_A = 4$

$V_C/V_D = V_B/V_A$

2)

	ΔU	W	Q
$A \to B$	0	$nRT_H \ln V_B/V_A$	$nRT_H \ln V_B/V_A$
$B \to C$	$\frac{3}{2}nR(T_c - T_H)$	0	$\frac{3}{2}nR(T_c - T_H)$
$C \to D$	0	$nRT_c \ln V_D/V_C$	$nRT_c \ln V_D/V_C$
$D \to A$	$\frac{3}{2}nR(T_H - T_c)$	0	$\frac{3}{2}nR(T_H - T_c)$

$$W_{TOTAL} = nRT_H \ln V_B/V_A + nRT_c \ln V_D/V_C$$
$$= nRT_H \ln V_B/V_A + nRT_c \ln V_A/V_B$$
$$= nRT_H \ln V_B/V_A - nRT_c \ln V_B/V_A$$
$$W_{Tot} = nR(T_H - T_c) \ln V_B/V_A$$

$$Q_{IN} = nRT_H \ln V_B/V_A + \tfrac{3}{2}nR(T_H - T_c) \qquad (\text{All } Q > 0)$$

$$\varepsilon = \frac{W_{TOT}}{Q_{in}} = \frac{\cancel{nR} \ln V_B/V_A (T_H - T_c)}{\cancel{nR} T_H \ln V_B/V_A + \tfrac{3}{2}\cancel{nR}(T_H - T_c)} = \frac{\ln V_B/V_A (T_H - T_c)}{T_H \ln V_B/V_A + \tfrac{3}{2}(T_H - T_c)}$$

$$\boxed{\varepsilon = \frac{(T_H - 300)\ln 4}{T_H \ln 4 + \tfrac{3}{2}(T_H - 300)}} = \frac{(T_H - 300)\ln 4}{T_H(\ln 4 + \tfrac{3}{2}) - 450}$$

3) Carnot is given by:

$$\varepsilon_c = \frac{T_H - T_C}{T_H} = \frac{(T_H - 300)}{T_H}$$

The numerator of ε is 1.4 larger than the Carnot, but the denominator is about 2.9 times larger (not counting) the -450 term. So at high temperatures this is about 48% less efficient than a Carnot. At lower T_H it gets closer to ε_c. This is not too large.

4) A chart:

T_H	ε_c	ε	$\varepsilon/\varepsilon_c$	$\varepsilon/\varepsilon_{max} = \varepsilon/.480$
350	.143	.124	86.5%	25.8%
400	.250	.197	78.7	41.0%
600	.500	.324	64.9%	67.5%
900	.667	.387	58.1%	80.6%
1200	.750	.414	55.2%	86.3%
2000	.850	.443	52.1%	92.3%

5) Notice how at high T_H, the ratio $\varepsilon/\varepsilon_c$ is approaching 48%
The ratio $\varepsilon/\varepsilon_c$ is 80% at ~ 400 K, $\varepsilon/\varepsilon_{max}$ is 80% at about 900 K.

6) From Chart and Table, a little less than 400 K.

7) $\frac{\text{money earned}}{\text{money spent}} = \$ = \frac{W \times 10^{-6}\ \text{dollars/Joule}}{(T_H)^4 \times 10^{-7}\ \text{dollars/Joule}\cdot\text{K}} = \frac{W\,c}{(T_H)^4\,c'}$

$$\boxed{\$ \ \frac{c\,n\,R(T_H - T_C)\ln V_B/V_A}{c'\,(T_H)^4}}$$

To find best T_H, take the derivative of this + set it to 0

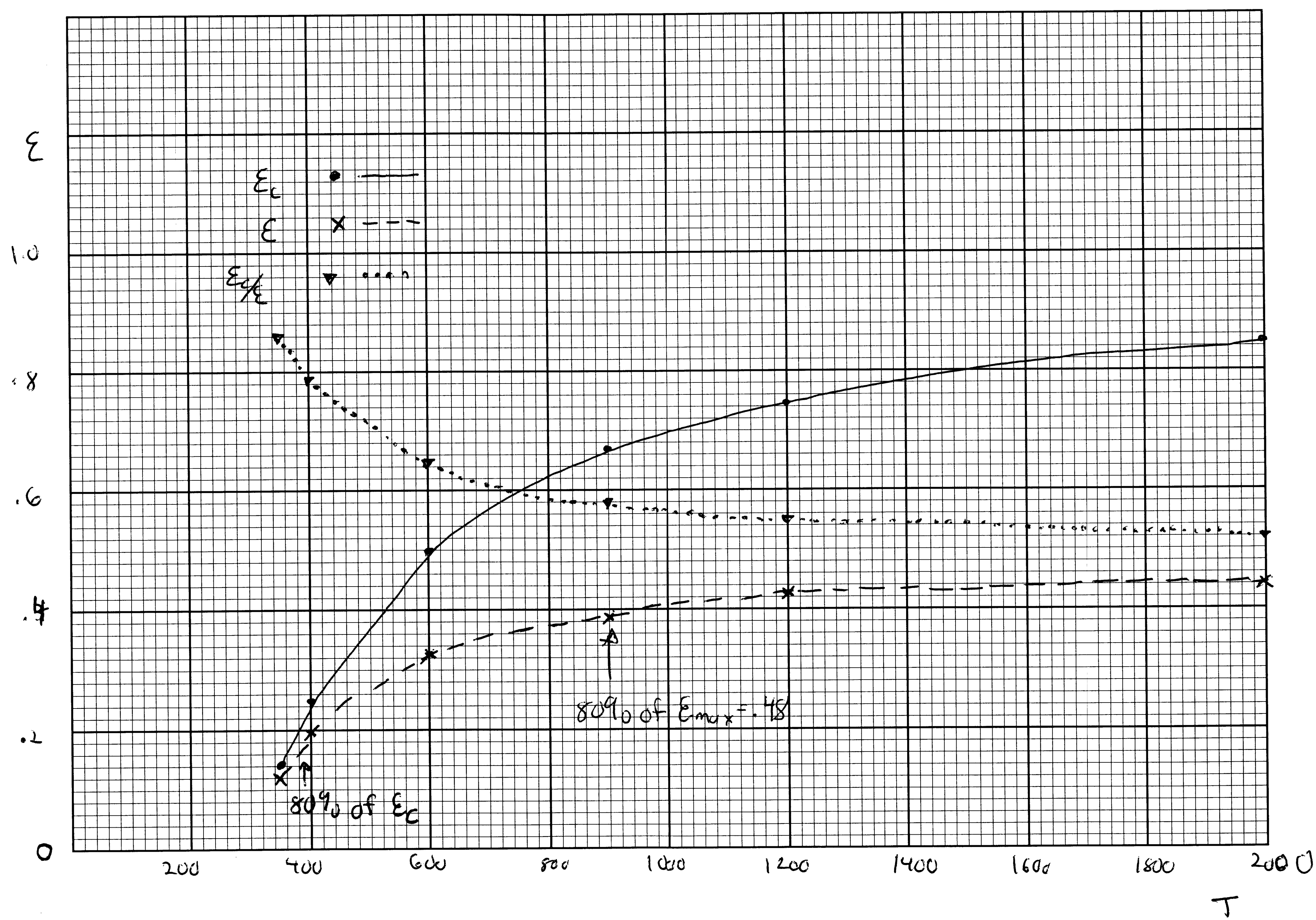
$$\frac{d\$}{dT_H} = \frac{d}{dT_H}\left(\frac{c\,n\,R\ln V_B/V_A}{c'\,T_H^3} - \frac{c\,n\,R\,T_C\ln V_B/V_A}{c'\,T_H^4}\right) = 0$$

$$= \frac{-3\,c\,n\,R\ln V_B/V_A}{c'\,T_H^4} + \frac{4\,c\,n\,R\,T_C\ln V_B/V_A}{c'\,T_H^5} = 0$$

$$\Rightarrow T_H = \frac{4}{3}T_C = \frac{4}{3}(300\text{ K}) = \boxed{400\text{ K}}$$

At 900 K $\$ = 1\times10^{-4}\,n$
At 400 K $\$ = 4.5\times10^{-4}\,n$
At 500 K $\$ = 3.7\times10^{-4}\,n$

8) I would recommend the insulation. They could run the engine at about 500 K and get the same efficiency as the current engine at 900 K. In doing so, they would have a profit margin 3.7 times higher than at 900 K.



ε
ε_c
ε
$\varepsilon_c/\varepsilon$
1.0
.8
.6
.4
.2
0
200
400
600
800
1000
1200
1400
1600
1800
2000
T
80% of ε_{max} = .48
80% of ε_c