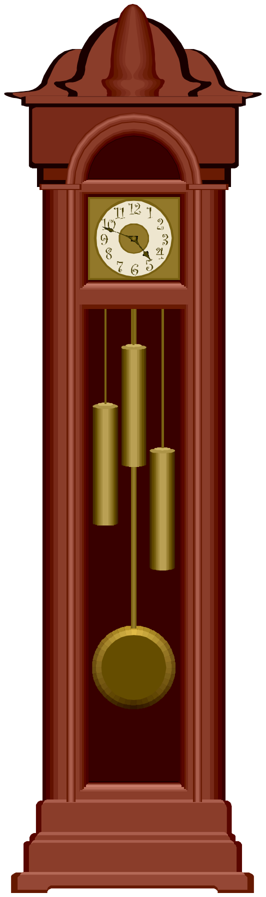
Chapter 7

Hooke's Force law and Simple Harmonic Oscillations



Hooke's Law

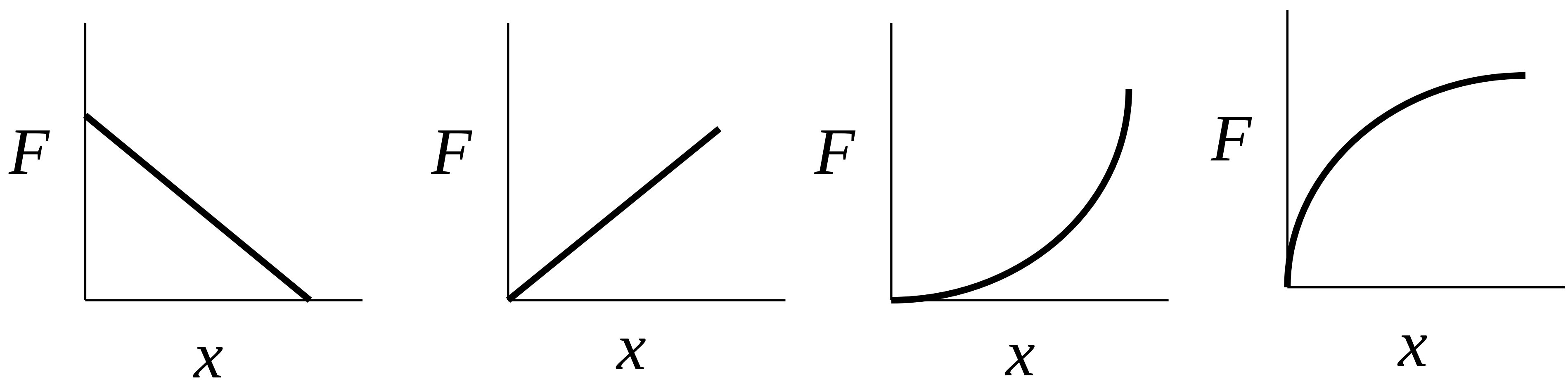
- An empirically derived relationship that approximately works for many materials over a limited range.
- Exactly true for a "massless, ideal spring."
- Hooke's law for an ideal spring is

$$F = -k(x - x_0)$$

where F is the force, x_0 is the equilibrium position of the spring, x is the distance the spring has been compressed/stretched and k is the "spring constant"

Interactive Question

The following plots show the value of a force directed along the *x* axis. Which plot is correct for an ideal spring?

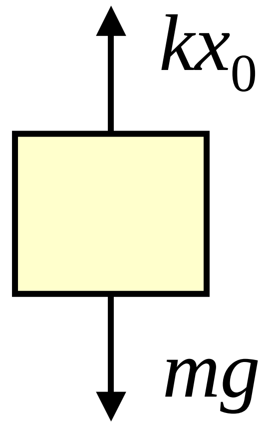


(A) (B) (C) (D)

(E) More than one of the above is correct.

Vertical Springs

At the equilibrium position



$kx_0 = mg$
$x_0 = mg/k$

Now move the spring an additional distance of x the forces on the mass are given by

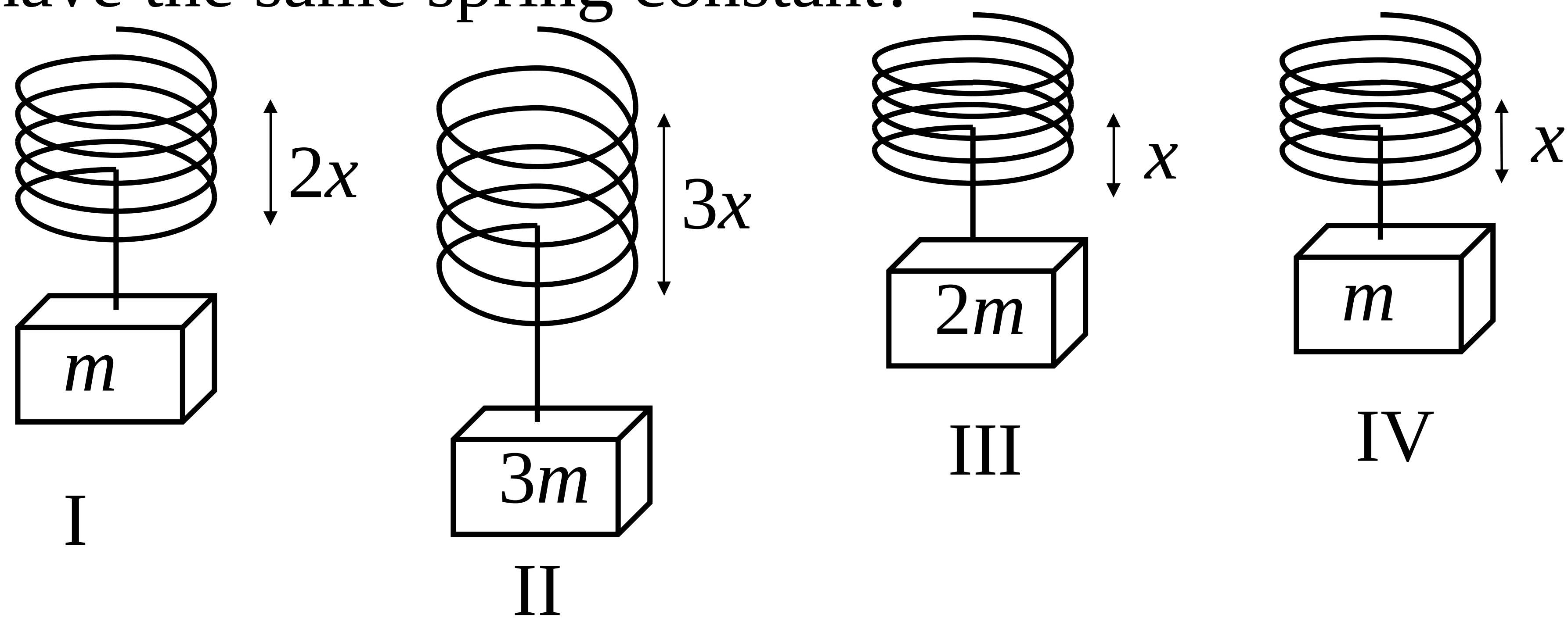
$$F = -k(x_0 + x) + mg = -k(mg/k + x) + mg = -kx$$

A vertical spring behaves as if it was horizontal with the restoring force equal to $-k\Delta x$, where Δx is the distance from the equilibrium position when the mass is simply hanging down from the spring.

Problem: At a grocery store, a spring on a scale is stretched by 2.3 cm when a bag of apples weighing 1.2 N is placed on the scale. What is the spring constant for the scale ?

Interactive Question

The following diagrams show the distance that a spring has been stretched from its equilibrium position when an object is hung from it. Which pair of springs have the same spring constant?



A) I & II B) III & IV C) II & IV

D) I & III E) None of the above

Simple Harmonic Motion

Periodic motion, or oscillatory motion, is motion that repeats itself. The simplest periodic motion to understand is called simple harmonic motion (SHM). It occurs when an object displaced from its equilibrium position feels a restoring force that is proportional to the distance from the equilibrium position. In other words, when the force is given by

$$F = -kx$$

This is exactly Hooke's law for springs. So ideal springs exhibit SHM. So let's consider an object attached to an ideal spring.

The motion of the object at the end of a spring repeats itself after a period of time, *T*, the time it takes to complete one cycle. The frequency (*f* or ν) is the number of cycles per unit time, so

$$\nu = 1/T$$

The SI unit of frequency is cycles/second.
1 cycle/second = 1 Hertz (Hz).

The maximum distance from the equilibrium point is called the amplitude (*A*), and the distance from the equilibrium point at any time is the displacement.

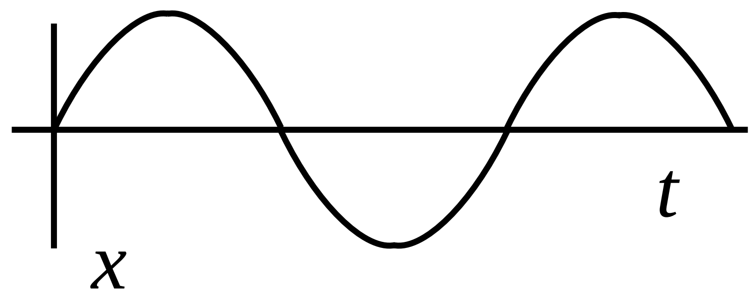
Interactive Question

Which of the following is necessary to make an object oscillate?

A) A disturbance around any equilibrium point.
B) A disturbance around a point of stable equilibrium.
C) Little or no friction
D) A and C
E) B and C

Interactive Question

An object is attached to an ideal spring and is oscillating. Consider two possibilities: (i) at some point during the oscillation the mass has zero velocity but is accelerating; (ii) at some point during the oscillation the mass has zero velocity and zero acceleration.



A) Both occur sometime during the oscillation

B) Neither occur during the oscillation

C) Only (i) occurs

D) Only (ii) occurs

Dynamics & Kinematics of Simple Harmonic Motion

$$F = -kx = ma = m\, d^2x/dt^2$$

$$d^2x/dt^2 + (k/m)x = 0$$

This is the equation of motion for a simple harmonic oscillator. It is a second order differential equation. The solution is a generalized sine or cosine and can be written as

$$x = A \cos(\omega t + \phi)$$

A: Maximum displacement, or amplitude

ω: Angular frequency

ϕ: is an arbitrary constant, depending on the initial conditions

$$x = A\cos(\omega t + \phi)$$

For instance, if $\phi = \pi/2$ then the cosine curve looks like a sine curve since

$\cos(\omega t+\pi/2) = \cos(\omega t)\cos(\pi/2) + \sin(\omega t)\sin(\pi/2) = \sin(\omega t)$

The cosine or sine repeats every 2π, so one period occurs when

$\omega T = 2\pi$

$T = 2\pi/\omega$

$\nu = \omega/(2\pi)$

And we see again that $\omega = 2\pi\nu$

Let's show that our result actually solves the equation of motion

$$d^2x/dt^2 + (k/m)x = 0$$

$x = A\cos(\omega t + \phi)$
$dx/dt = v = -A\omega\sin(\omega t + \phi)$
$d^2x/dt^2 = a = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x$

$$d^2x/dt^2 + (k/m)x = -\omega^2 x + (k/m)x = 0$$

So our solution only works when $\omega^2 = k/m$,

$$\omega = \sqrt{\frac{k}{m}}$$

The frequency of oscillation only depends on the spring constant (stiffness), and the mass.

$$v = dx/dt = -A\omega \sin(\omega t + \phi)$$
$$a = d^2x/dt^2 = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x$$

The maximum of these quantities is given when the sine or cosine is equal to 1, so $v_{max} = A\omega$ and $a_{max} = A\omega^2$

Simple harmonic motion occurs if and only if

1. The force is restorative and proportional to the distance
2. The equation of motion is given by $d^2x/dt^2 + \omega^2 x = 0$

These two criteria are really identical.

Interactive Question

In simple harmonic motion

A) The velocity is greatest at the maximum displacement
B) The period depends on the amplitude
C) The acceleration is constant
D) The acceleration is greatest at zero displacement
E) The acceleration is greatest at the maximum displacement

Interactive Question

A mass is attached to an ideal spring. When it is stretched a distance x, the system vibrates with a frequency ν. In order to increase the frequency, one would have to

A) reduce the spring constant.
B) increase the length of the spring.
C) decrease the mass on the end of the spring.
D) reduce the distance that the spring is initially stretched.
E) increase the distance that the spring is initially stretched.

Interactive Question

A particle is in simple harmonic motion with period T. At time $t = 0$, it is at the equilibrium point. At which of the following times is it furthest from the equilibrium point?

A) $t = T/2$
B) $t = 3T/4$
C) $t = T$
D) $t = 3T/2$
E) None of the above

Interactive Question

A particle is in simple harmonic motion with period T. At time $t = 0$, it is halfway between the equilibrium point and an end point of its motion, traveling toward the end point. The next time it is at the same place is

A) $t = T$
B) $t = T/2$
C) $t = T/4$
D) $t = T/8$
E) None of the above

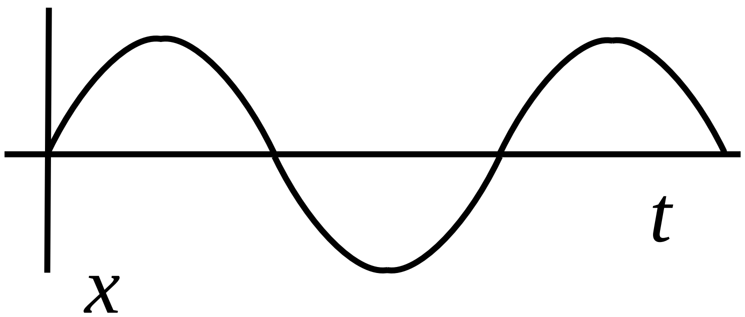
Interactive Question

The displacement of a mass oscillating on a spring is given by $x = A\cos(\omega t + \phi)$. If the initial displacement is zero and the initial velocity is in the negative x diretion, then the phase constant ϕ is:

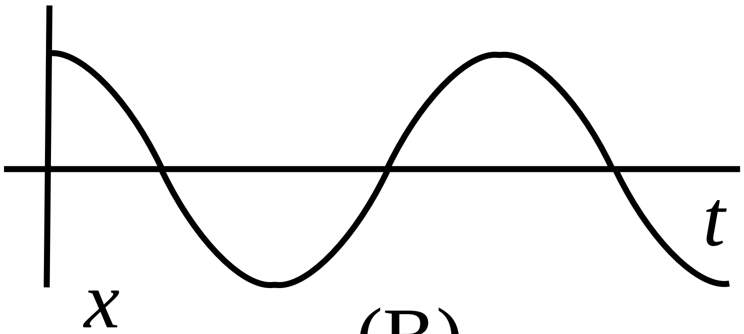
A) 0
B) $\pi/2$
C) π
D) $3\pi/2$
E) 2π

Interactive Question

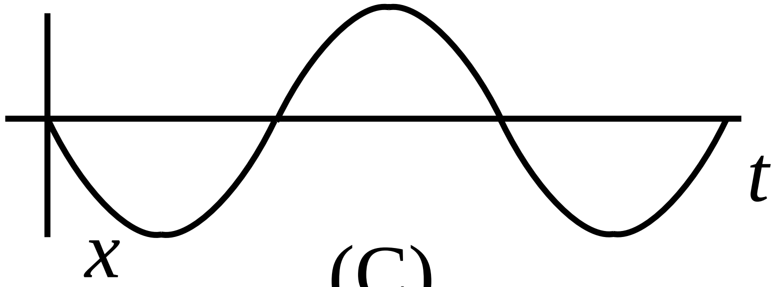
The simple harmonic motion of an object attached to an ideal spring is given by $x = A \sin(\omega t + \phi)$. Which of the following plots has a phase $\phi = -\pi/2$?



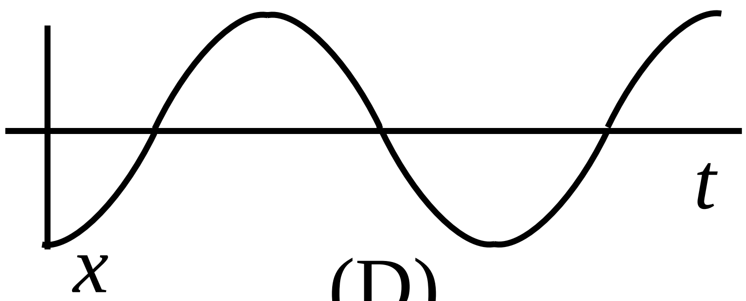
(A)



(B)

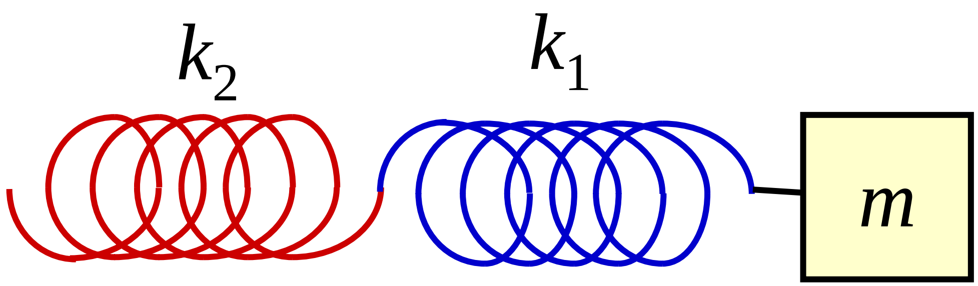


(C)



(D)

Problem: A block of m = 1.2 kg sitting on an air track is attached to a spring with a spring constant k_1 = 450 N/m which is attached to another spring of spring constant k_2 = 220 N/m. When the springs are stretched a little, what is the period of oscillation for this combination?



Problem: An object with a mass of 0.35 kg object is attached to a spring with spring constant 13 N/m and is free to move on a frictionless horizontal surface. The equilibrium position is at $x = 0$. If the object started with a velocity of 0 m/s at a position of $x = 0.23$

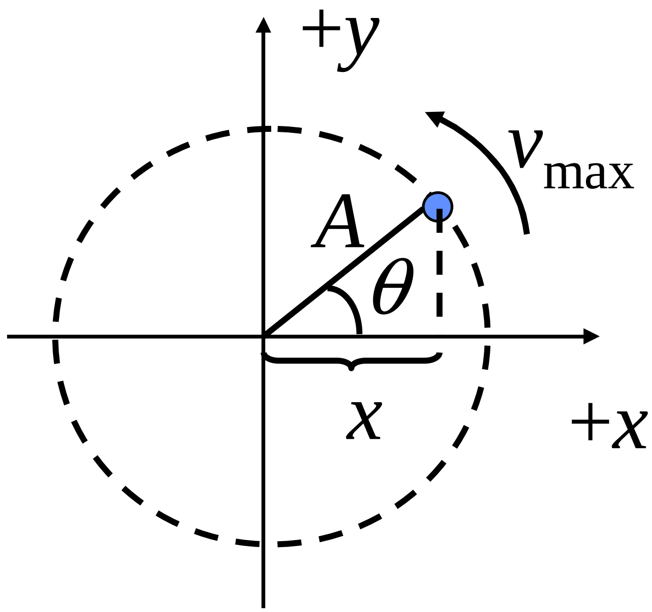
(a) Find the force on it and its acceleration at $x = -0.10$ m,

(b) Find the force on it and its acceleration at $x = 0$ m

(c) Find the phase, δ.

(d) Find the equation for the x position at all times.

Relationship of SHM to Circular Motion



The position (and velocity) along the *x* axis is given by

$$x = A\cos\theta = A\cos\omega t$$
$$v = dx/dt \;\; -A\omega\sin\omega t$$

which is the same result we got for the velocity of SHM.

We can write this in the same form as the equation for velocity that we derived using energy principles.
From rotational motion, we know that $v_{max} = A\omega$.
Using geometry, we see that $\sin\theta = (A^2 - x^2)^{1/2}/A$
$v = -v_{max}(A^2 - x^2)^{1/2}/A \;\; = -v_{max}(1 - x^2/A^2)^{1/2}$

It is sometimes easier to understand SHM using this idea of a reference circle. For instance, the speed of the ball going around the circle is given by distance divided by time.

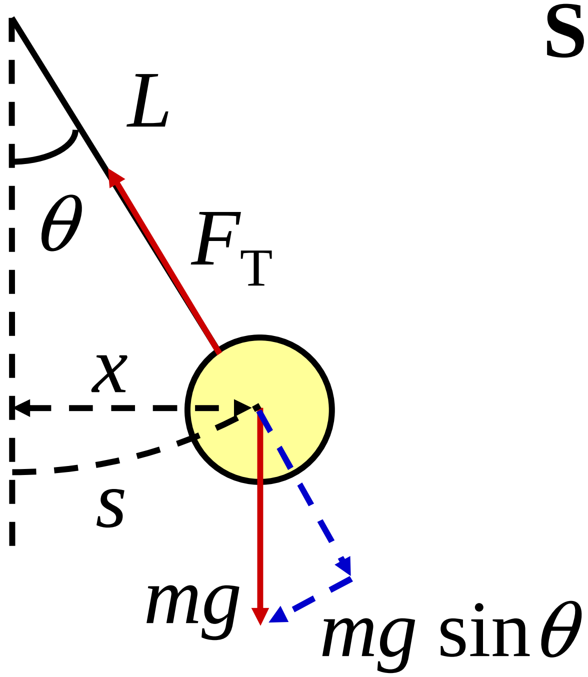
$v_{\max} = (2\pi A)/T$

$T = (2\pi A)/v_{\max}$

Since $v_{\max}^2 = (k/m)A^2$

$T = (2\pi A) \div \{(k/m)^{1/2}A\} = 2\pi\,(k/m)^{1/2} = 2\pi/\omega$

Simple Pendulum



Along the tangential (s) direction

$$\sum F_t = ma$$

$$-mg \sin\theta = m\, d^2s/dt^2$$

$$-mgx/L = m\, d^2s/dt^2$$

For SHM to occur, the force must be proportional to the distance traveled. In this case, the force is proportional to x, but the distance traveled is in the s direction.

When the angle is small, though, $x \approx s$:

$$-mgx/L = m\, d^2x/dt^2$$

$$-mgx/L = m\, d^2x/dt^2$$

This force looks just like Hooke's law, $F = -kx$ with $k = mg/L$

Then the pendulum executes simple harmonic motion with

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}}$$

$$\omega = 2\pi\nu = \frac{2\pi}{T} = \sqrt{\frac{g}{L}}$$

Interactive Question

Simple pendulum *A* swings back and forth at twice the frequency of simple pendulum *B*. Which statement is correct?

A) Pendulum *B* is twice as long as pendulum *A*.

B) Pendulum *B* is twice as massive as pendulum *A*.

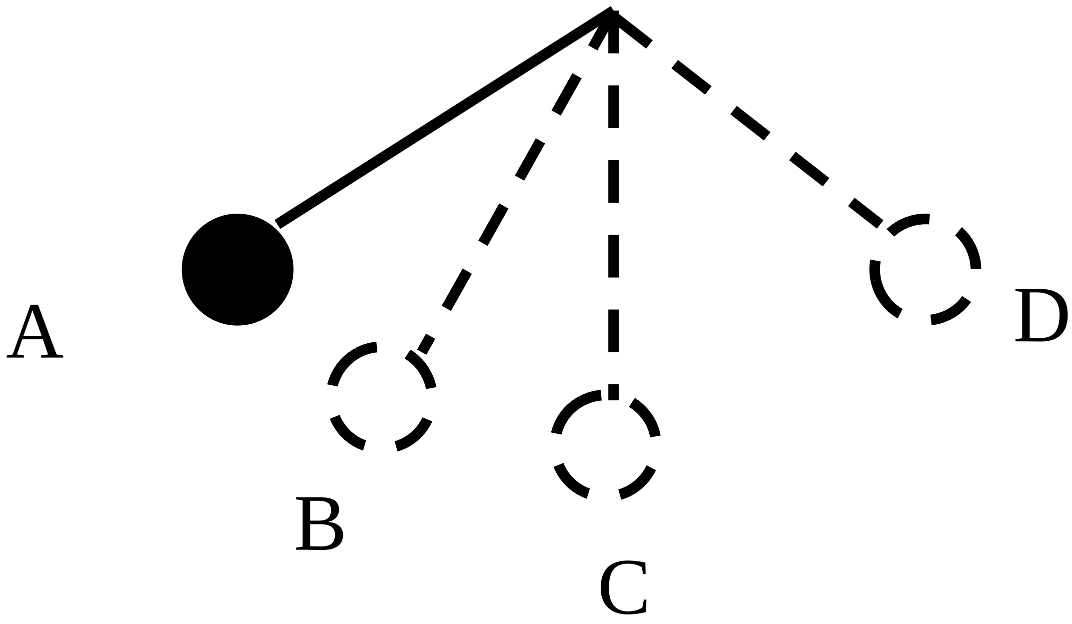
C) The length of *B* is four times the length of *A*.

D) The mass of *B* is four times the mass of *A*.

E) The length of *B* is half the length of *A*.

Interactive Question

A mass on the end of a massless string undergoes simple harmonic motion. Where is the instantaneous acceleration of the mass greatest?



A) A
B) B
C) C
D) A and C
E) A and D

Interactive Question

A simple pendulum of length 1 m has a period of roughly 2 s on the surface of the earth. If the same pendulum were placed in a satellite that orbits the earth at an altitude of one earth radius,

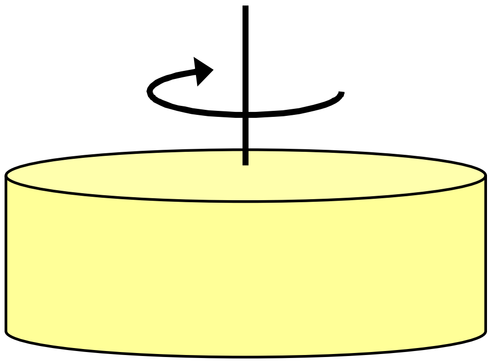
A) its period is unchanged.
B) its period is half as large.
C) its period is twice as large.
D) its period would decrease to 0.
E) its period would be infinitely as large.

Problem: A man wants to know the height of a building which has a pendulum hanging from its ceiling. He notices that in one minute the pendulum oscillates 8 times.

(a) What is the height of the building?

(b) If the length were cut in half, what would the new frequency be?

The Torsion Pendulum



The restoring force is proportional to the angle through which the pendulum has been twisted.

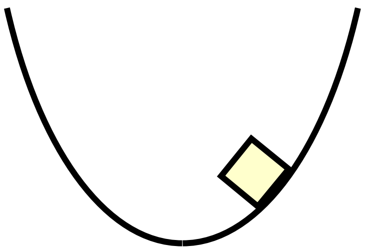
$$\tau = -\kappa\theta = I\, d^2\theta/dt^2$$

The period can be directly written:

$$T = 2\pi\sqrt{I/\kappa}$$

Interactive Question

A small block is placed on a frictionless ramp that is in the shape of a parabola. (The equation for a parabola centered at $x = 0$ is given by $y = Cx^2$ where C is a constant.) When released, will this block exhibit simple harmonic motion?



A) No
B) Yes, but only for small oscillations
C) Yes, for all oscillations

Problem: One atom of a diatomic molecule feels a force given by $F = -C/r^2 + D/r^3$ where C and D are positive constants.

A) At what point does equilibrium occur?

B) Will this atom exhibit simple harmonic motion for small oscillations?

C) What is the force constant?

D) What is the period of the motion?

Damped Oscillations

Most oscillations do not continue on forever, but eventually stop, due to some kind of nonconservative force like friction or air resistance. Some vibrations are purposely stopped. The shock absorbers in your car are made to stop the vibrations set up by the road. All these vibrations which are eventually stopped are called damped vibrations, or if the harmonic motion is stopped, it is called damped harmonic motion. When the motion is damped, mechanical energy is not conserved. In many cases, the damping motion is proportional to the velocity.

Damped Oscillations

$\sum F = ma$

$-kx - bv = ma$

$kx + b(dx/dt) + m(d^2x/dt^2) = 0$

The solution to this equation is

$$x = A_0\, e^{-bt/2m} \cos(\omega' t + \delta)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \omega_0 \sqrt{1 - \left(\frac{b}{2m\omega_0}\right)^2}$$

You can verify this by plugging the solution back into the original equation.

Damped Oscillations

$$x = A_0\, e^{-bt/2m} \cos(\omega' t + \delta)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \omega_0 \sqrt{1 - \left(\frac{b}{2m\omega_0}\right)^2} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$b = 0$: No damping.

Small b ($b^2 < 4mk$): underdamping

Large b ($b^2 > 4mk$): overdamping. The square root is negative. There are no oscillations at all.

$b^2 = 4mk$: critical damping. The minimum time for oscillations to cease

Resonance and Driven Oscillations

Everything has a natural frequency (or frequencies) that it will oscillate at. These are called resonance frequencies.

Let's apply an external force on an object:

$$F_{\text{ext}} = F_0 \cos \omega t$$

$$\sum F = ma$$

$$F_0 \cos \omega t + kx + b(dx/dt) = m(d^2x/dt^2)$$

The solution, after a long time, is given by

$$x = A \cos(\omega t + \phi)$$

$$\phi = \tan^{-1}\left(\frac{b\omega}{\omega_0^2 - \omega^2}\right) \qquad A = \frac{F_0}{\sqrt{m^2\left(\omega^2 - \omega_0^2\right)^2 + b^2\omega^2}}$$

Interactive Question

Resonance occurs in harmonic motion when

A) the system is overdamped.
B) the system is critically damped.
C) the energy in the system is a minimum.
D) the driving frequency is the same as the natural frequency of the system.
E) the energy in the system is proportional to the square of the motion’s amplitude.

Interactive Question

An oscillator is subject to a damping force that is proportional to its velocity and an external sinusoidal force that is applied. After a long time:

A) its amplitude is an increasing function of time.

B) its amplitude is a decreasing function of time.

C) its amplitude is constant.

D) its amplitude is a decreasing function of time only if the damping constant is large.

E) its amplitude increases over some portions of a cycle and decreases over other portions.