# Chapter 2

# A Mathematical Toolbox

# Vectors and Scalars

1) Scalars have only a magnitude (numerical value) Denoted by a symbol, *a*
2) Vectors have a magnitude and direction
   - Denoted by a bold symbol (**A**), or symbol with an arrow ($\vec{A}$)
   - The magnitude of the vector **A** is the scalar, *A*.
   - Vectors are drawn as an arrow where the length of the arrow is proportional to the magnitude and the direction of the arrow gives the direction of the vector.
   - Vectors do not have a specific location. They can be moved around as long as they maintain their magnitude and direction.

# Interactive Question

Which of the following is a vector quantity?

A) The age of the earth.
B) The mass of a football.
C) The earth's pull on your body.
D) The temperature of an iron bar.
E) The number of people attending an OU football game.

# Vectors

Vector quantities have a magnitude and direction.

- A quantity that is a vector must be given as two ( or three) independent pieces of information in two (or three) dimensions.

1. Magnitude and direction (two angles in 3-d)

or

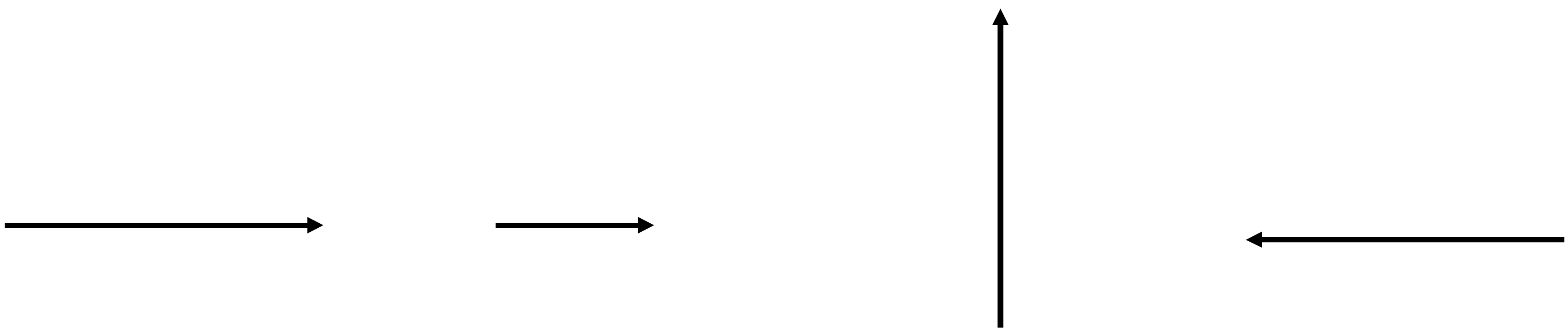
2. Components along each direction.

# Interactive Question

The vectors **A** is shown.

**A**

Which vector(s) below is equal to 2**A**?



(A) (B) (C) (D)

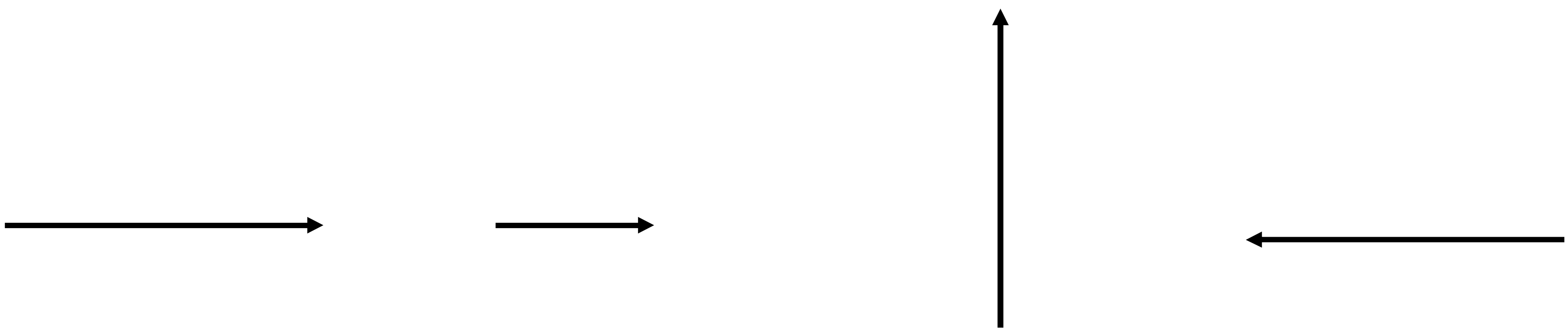
(E) More than one of the above is equal to 2**A**

# Interactive Question

The vectors **A** is shown.

**A**

Which vector(s) below is equal to –2**A**?



(A) (B) (C) (D)

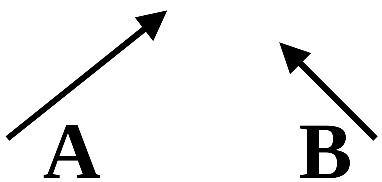
(E) More than one of the above is equal to –2**A**

# Addition and Subtraction of Vectors

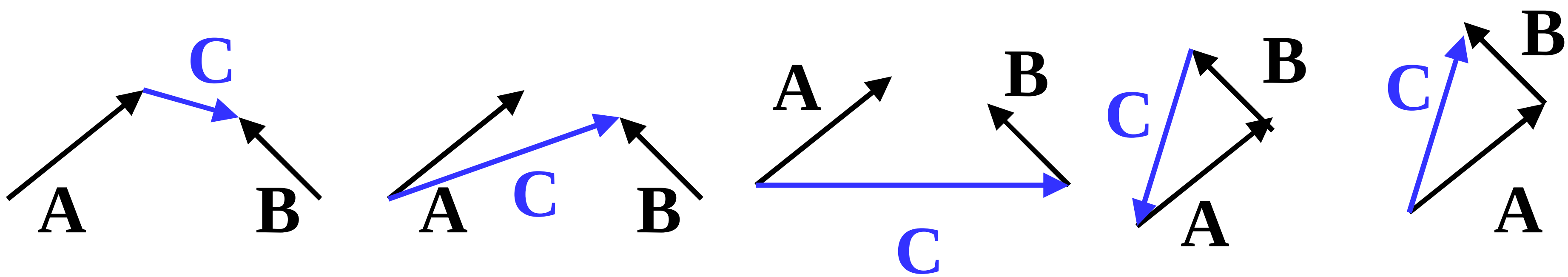
Vectors are not added or subtracted like scalars. To add two vectors graphically, put the tail of one vector on the tip of the other and draw the resultant. To subtract two vectors, you simply add the negative of one vector to the other vector, so that $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$

# Interactive Question

Vectors **A** and **B** are shown below.



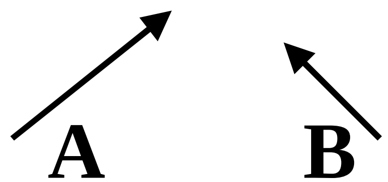

Which diagram below correctly shows the vector **C**, where **C** = **A** + **B**



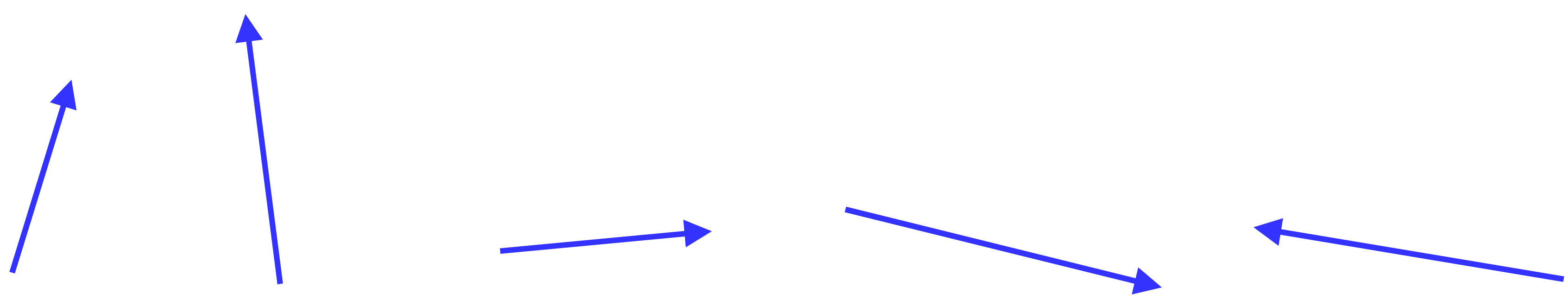

(A) (B) (C) (D) (E)

# Interactive Question

Vectors **A** and **B** are shown below.



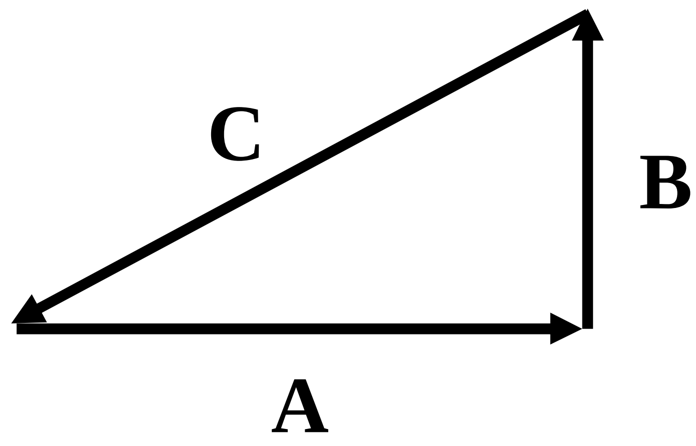

Which diagram best shows the vector **C**, where
**C** = **A** – 2**B**



(A) (B) (C) (D) (E)

# Interactive Question

Which expression is not true concerning the vectors show in the sketch at the right?



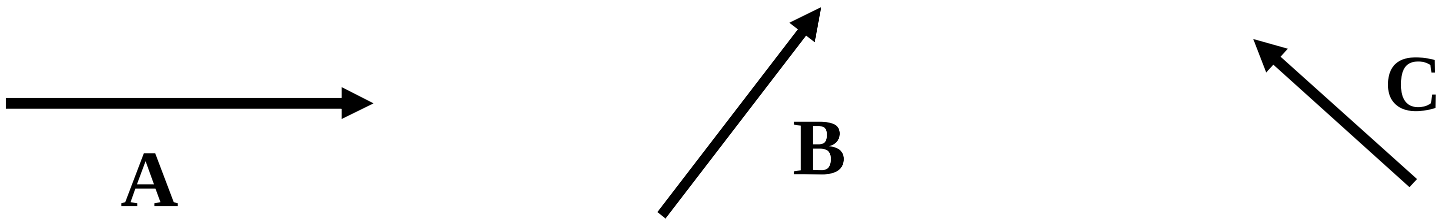

A) $\mathbf{C} = \mathbf{A} + \mathbf{B}$

B) $\mathbf{C} + \mathbf{A} = -\mathbf{B}$

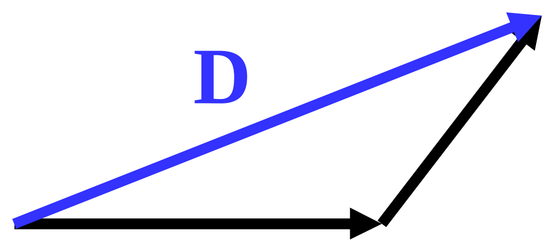
C) $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$

D) $C < A + B$

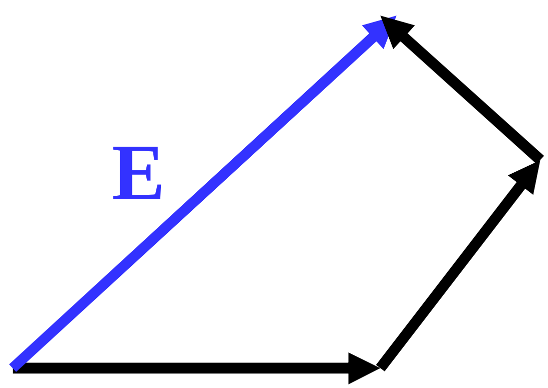
E) $A^2 + B^2 = C^2$



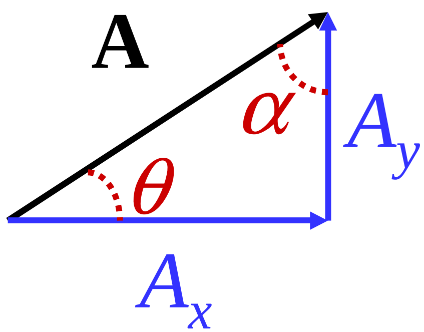

Draw the vector, **D**, such that **D** = **A** + **B**

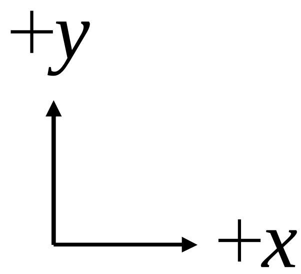



Draw the vector, **E**, such that **E** = **A** + **B** + **C**



# Vector Components

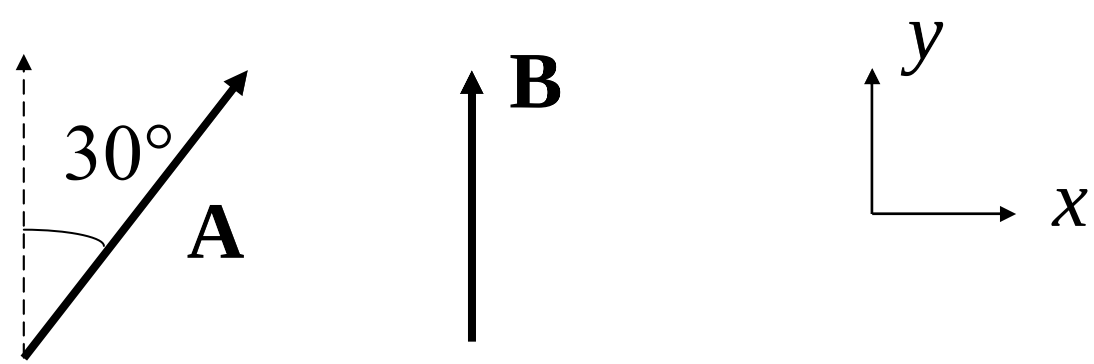



$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$$
$$A^2 = {A_x}^2 + {A_y}^2$$
$$\cos\theta = A_x/A, \;\; \sin\theta = A_y/A$$
$$A_x = A\cos\theta = A\sin\alpha$$
$$A_y = A\sin\theta = A\cos\alpha$$

# Interactive Question

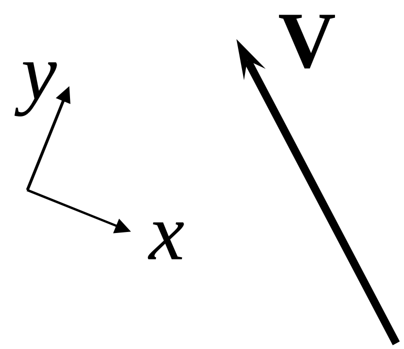
Two vectors, **A** and **B** are shown below. Which expressions gives the correct value for the *x* component of **A** and **B?**



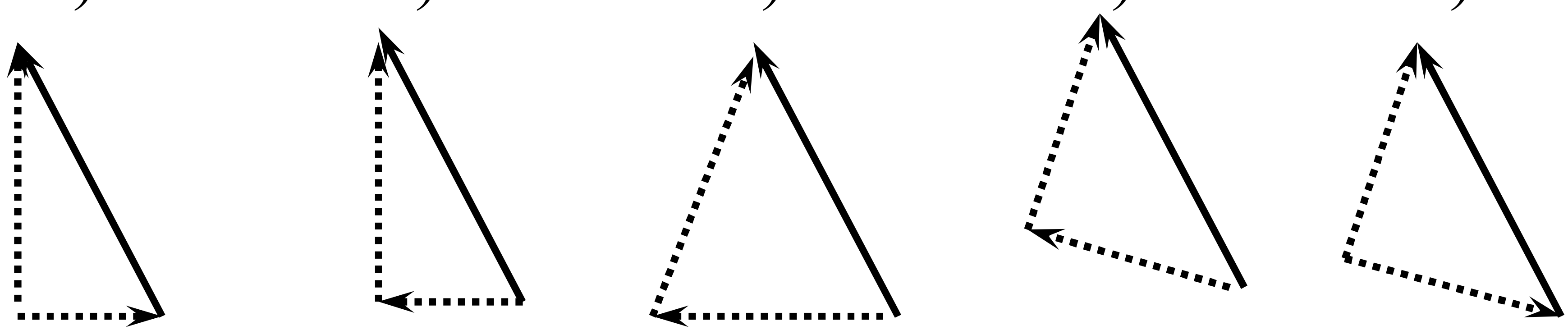

| | $x$ component of **A** | $x$ component of **B** |
|---|---|---|
| A) | $A \cos 30°$ | $B \cos 0°$ |
| B) | $A \cos 30°$ | $B$ |
| C) | $A \cos 30°$ | 0 |
| D) | $A \sin 30°$ | $B \cos 0°$ |
| E) | $A \sin 30°$ | 0 |

# Interactive Question

The vector **V** is shown at the right. The *x* and *y* axes are defined at an angle as shown, rather than in the vertical and horizontal directions. Which diagram shows the components of **V** in the *x* and *y* directions?

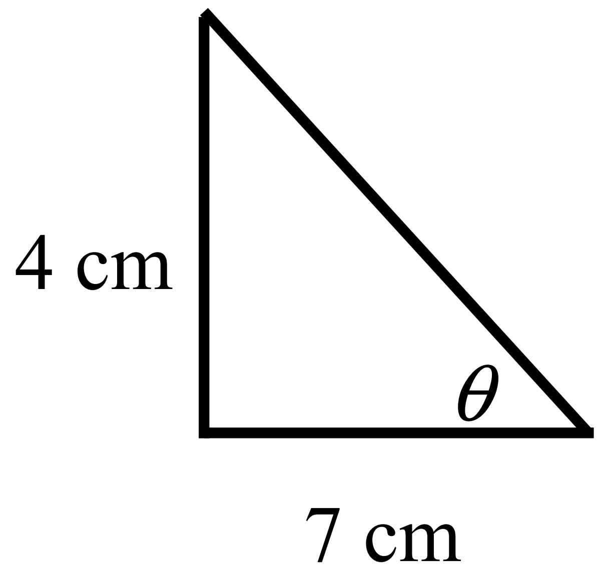



A) B) C) D) E)



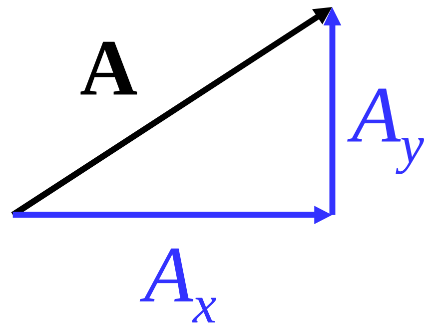
# Interactive Question

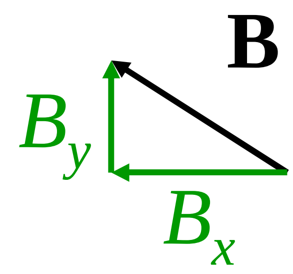
The figure at the right shows a right triangle. Which expression determines the angle $\theta$?



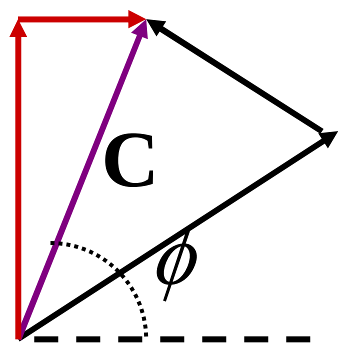

A) $\cos^{-1}(7/4)$
B) $\sin^{-1}(4/7)$
C) $\sin^{-1}(7/4)$
D) $\tan^{-1}(7/4)$
E) $\tan^{-1}(4/7)$

# Adding with Components

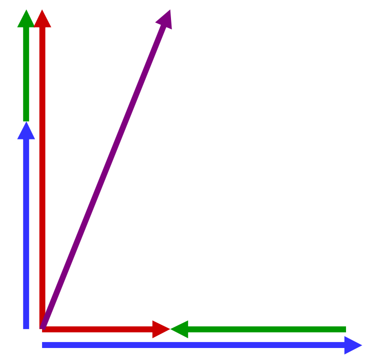



Find **C** = **A** + **B**




$$C_y = A_y + B_y$$



$$C_x = A_x + B_x$$

$$C = (C_x{}^2 + C_y{}^2)^{1/2}$$

$$\phi = \tan^{-1}(C_y/C_x)$$

# Unit Vectors

A unit vector has a magnitude of 1, and points in a particular direction. We usually define three unit vectors pointing along the *x*, *y*, and *z* axes and give them the names **i**, **j**, and **k**, respectively (sometimes written with "hats" to remind us they are unit vectors). We can add these vectors and multiply them by a scalar, just like other vectors.

Example: A plane is traveling with a velocity vector **v** that has components of 110 m/s in the *x* direction, –65 m/s in the *y* direction, and 35 m/s in the *z* direction. How is this written in terms of unit vectors?

$\mathbf{v} = (110\ \text{m/s})\mathbf{i} - (65\ \text{m/s})\mathbf{j} + (35\ \text{m/s})\mathbf{k}$

# Vector Equality

If **A** = **B**

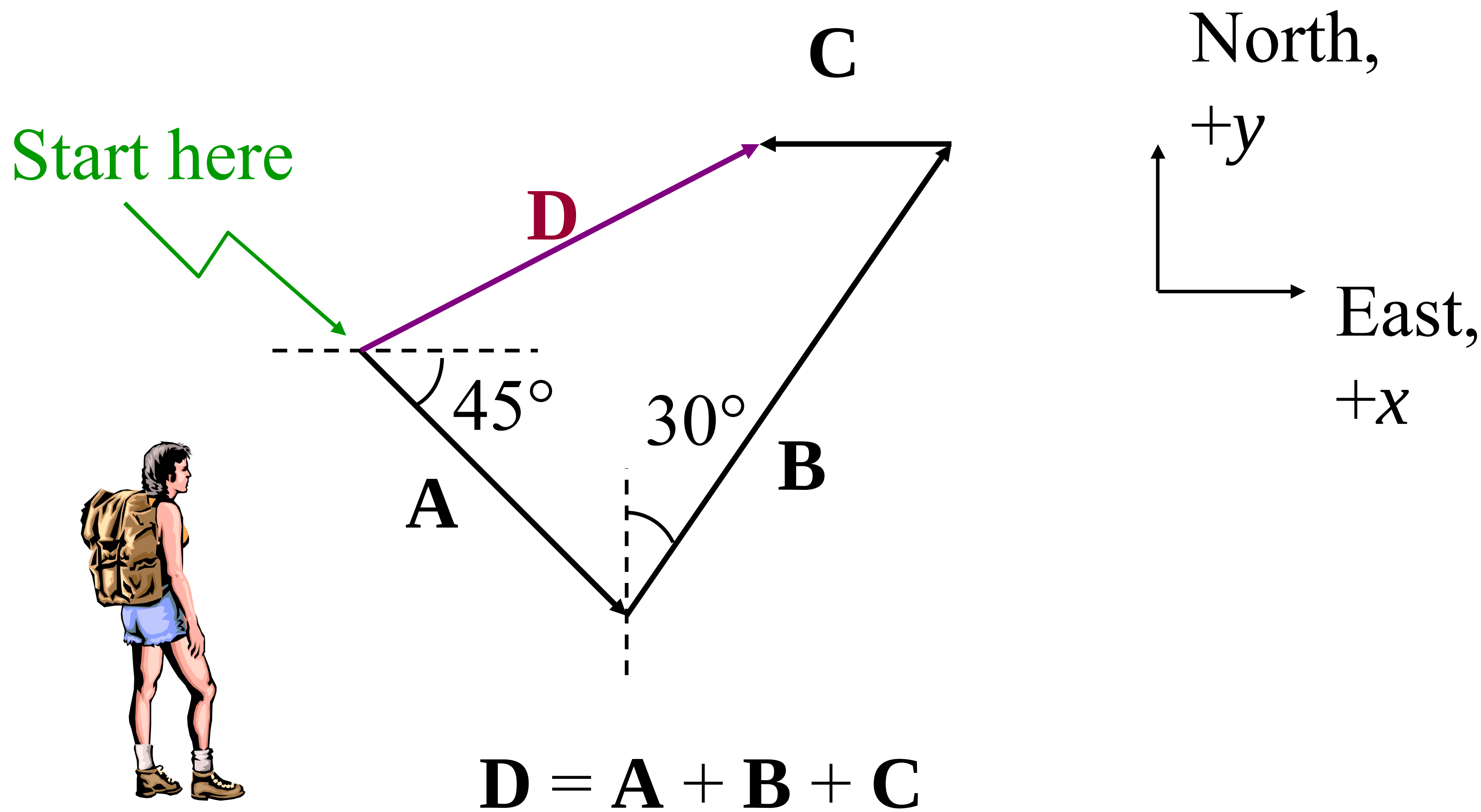
- $A_x\mathbf{i} + A_y\mathbf{j} + A_Z\mathbf{k} = B_x\mathbf{i} + B_y\mathbf{j} + B_Z\mathbf{k}$
- $A_x = B_x$, $A_y = B_y$, $A_z = B_z$

Vector notation is a short hand way of writing three simultaneous equations.

# Solving Problems with Vectors

1. Draw and label the direction of the *x* and *y* axes.
2. Draw and label all the vectors.
3. Determine the individual components of each vector.
   - Draw and label the components.
   - The components of the vector must be along the *x* and *y* axis, have magnitudes less than the original vector, and add up to equal the original vector.
   - Use trigonometry to find the length of the components.
   - The sign of the component is given by the direction of the component's arrow.
4. Do calculations in the *x* and *y* directions separately using the *x* and *y* components of the vectors, respectively. Or use vector notation and unit vectors to add components.
5. Combine the results from the *x* and *y* directions to get the final vector(s).

<u>Problem:</u> A hiker walks 5.0 km due southeast, then 8.0 km in a direction 30° east of north, then 2.0 km directly west. Where is she compared with where she started?



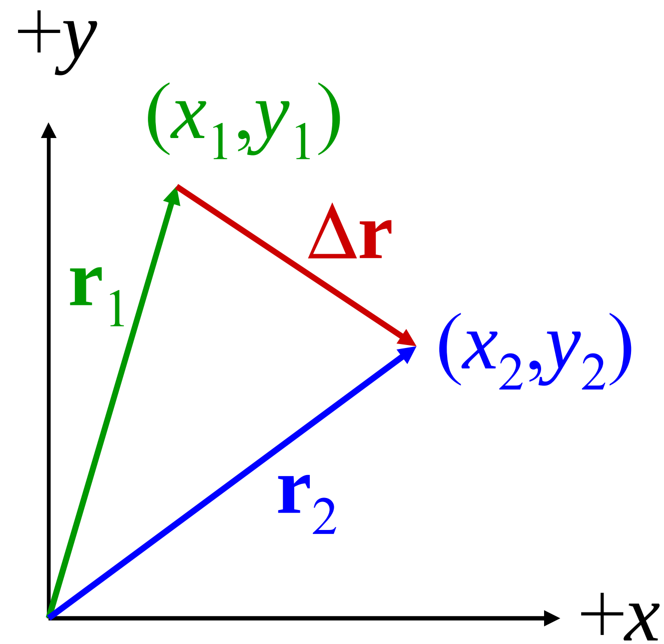

$$\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C}$$

# Multi-Dimensional Variables and Differences

An object that is at the point $(x_1, y_1, z_1)$ can be described as being at the point given by the position vector $\mathbf{r}_1$,

$$\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}.$$

If the object moves to a new position, $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, then its change in position is given by $\Delta\mathbf{r}$, (which is also a vector), with




Note that $\mathbf{r}_1 + \Delta\mathbf{r} = \mathbf{r}_2$

$$\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

For any vector **V**, we use the notation,

$$\Delta\mathbf{V} = \mathbf{V}_2 - \mathbf{V}_1 = (V_{2x} - V_{1x})\mathbf{i} + (V_{2y} - V_{1y})\mathbf{j} + (V_{2z} - V_{1z})\mathbf{k}$$

where $\Delta\mathbf{V}$ is itself a vector.

# Multiplying Vectors Together

Two ways to multiply vectors together:

1. Scalar product (or dot product) produces a scalar.
2. Vector product (or cross product) produces a vector.

# Scalar ("Dot") Product

$$\mathbf{A}\cdot\mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$
$$= (A_x\mathbf{i}\cdot B_x\mathbf{i} + A_x\mathbf{i}\cdot B_y\mathbf{j} + A_x\mathbf{i}\cdot B_z\mathbf{k}) + (A_y\mathbf{j}\cdot B_x\mathbf{i} +$$
$$A_y\mathbf{j}\cdot B_y\mathbf{j} + A_y\mathbf{j}\cdot B_z\mathbf{k}) + (A_z\mathbf{k}\cdot B_x\mathbf{i} + A_z\mathbf{k}\cdot B_y\mathbf{j} + A_z\mathbf{k}\cdot B_z\mathbf{k})$$
$$= A_xB_x + A_yB_y + A_zB_z = AB\cos\theta$$

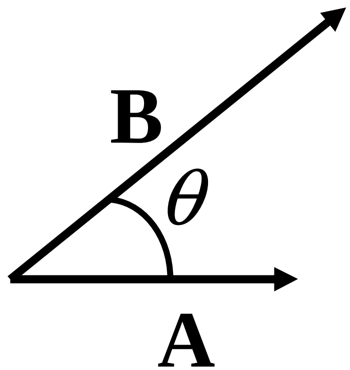
- $\theta$ is the angle between the vectors **A** and **B** when they are placed tail to tail
- $A_x$, $A_y$, and $A_z$ are the components of **A** along the $x$, $y$, and $z$ axes, respectively, (the same for **B**)
- $A$ and $B$ are the magnitudes of the vectors **A** and **B**

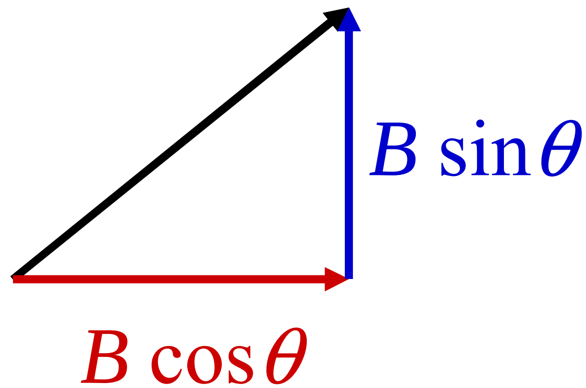
$$\mathbf{A}\cdot\mathbf{A} = A_xA_x + A_yA_y + A_zA_z = A_x^2 + A_y^2 + A_z^2 = A^2$$
$$A = \sqrt{\mathbf{A}\cdot\mathbf{A}}$$

# Properties of scalar product

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$

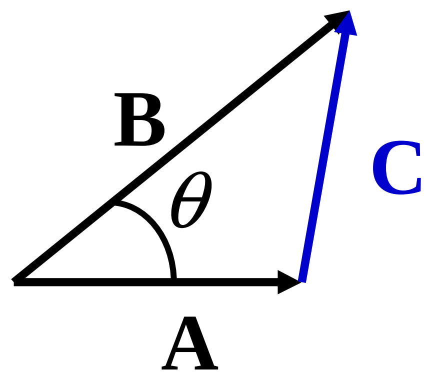
$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$




So we see that **A**·**B** is the magnitude of **B** in the direction of **A** multiplied by the magnitude of **A**, (or completely equivalently, the magnitude of **A** in the direction of **B** multiplied by the magnitude of **B**.)

$$\mathbf{i}\cdot\mathbf{i} = 1 \qquad\qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = 0$$




From the law of cosines for a triangle

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$

$$2AB\cos\theta = A^2 + B^2 - C^2$$

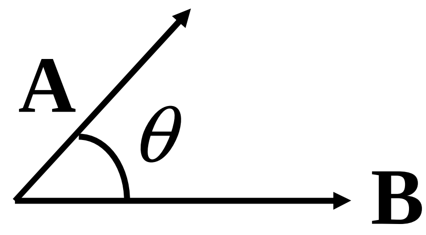
$$\begin{aligned}
2AB\cos\theta &= A^2 + B^2 - (\mathbf{B} - \mathbf{A})^2 \\
&= A_x{}^2 + A_y{}^2 + A_z{}^2 + B_x{}^2 + B_y{}^2 + B_z{}^2 - \\
&\quad \{(B_x - A_x)\mathbf{i} + (B_y - A_y)\mathbf{j} + (B_z - A_z)\mathbf{k}\}^2 \\
&= A_x{}^2 + A_y{}^2 + A_z{}^2 + B_x{}^2 + B_y{}^2 + B_z{}^2 - \\
&\quad \{(B_x{}^2 + A_x{}^2 - 2A_xB_x) + (B_y{}^2 + A_y{}^2 - 2A_yB_y) + \\
&\quad (B_z{}^2 + A_z{}^2 - 2A_zB_z\} \\
&= 2A_xB_x + 2A_yB_y + 2A_zB_z
\end{aligned}$$

$$AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

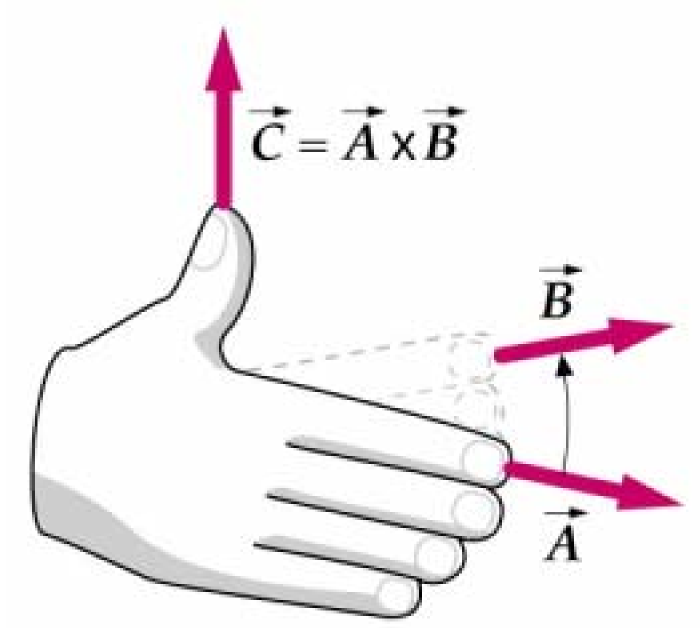
Problem: If $\mathbf{A} = -2.5\mathbf{i} + 3.2\mathbf{j} - 1.7\mathbf{k}$ and $\mathbf{B} = 4.2\mathbf{i} - 0.5\mathbf{j}$, what is the angle between **A** and **B**?

# Vector (Cross) Product

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$




$$C = AB \sin\theta$$




The direction of **C** is given by the right hand rule described in the text.

$$\mathbf{A} \times \mathbf{B} = -\,\mathbf{B} \times \mathbf{A}$$
$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k},\ \ \mathbf{j} \times \mathbf{k} = \mathbf{i},\ \ \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

We can write the components of the cross product as:

$$C_x = A_y B_z - A_z B_y$$
$$C_y = A_z B_x - A_x B_z$$
$$C_z = A_x B_y - A_y B_x$$

This can be easily remembered by using the determinant form:

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

# Interactive Question

If **A** points to the east (➔) and **B** points southwest (⬋), in what direction does **A**×**B** point?

A) Upward (out of the page)
B) Downward (in to the page)
C) North (⬆)
D) Northwest (⬉)
E) Southeast (⬊)

Problem: $\mathbf{A} = -2.5\mathbf{i} + 3.2\mathbf{j} - 1.7\mathbf{k}$ and $\mathbf{B} = 4.2\mathbf{i} - 0.5\mathbf{j}$.
What is $\mathbf{A} \times \mathbf{B}$?

# Some Vector Calculus

$$\frac{d\mathbf{A}}{dt} = \frac{dA_x}{dt} + \frac{dA_y}{dt} + \frac{dA_z}{dt}$$

$$\frac{d\left(\mathbf{A}\cdot\mathbf{B}\right)}{dt} = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$$

$$\frac{d\left(\mathbf{A}\times\mathbf{B}\right)}{dt} = \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}$$