Chapter 10

Spin and Orbital Motion

Some Definitions

Orbital motion: Motion relative to a point, often periodic, but not necessarily so.

Spin motion: Motion of an object as it rotates around an axis through its center of mass.

Rotational motion: Motion around an axis of rotation. Both orbital and spin motion are examples of rotational motion.

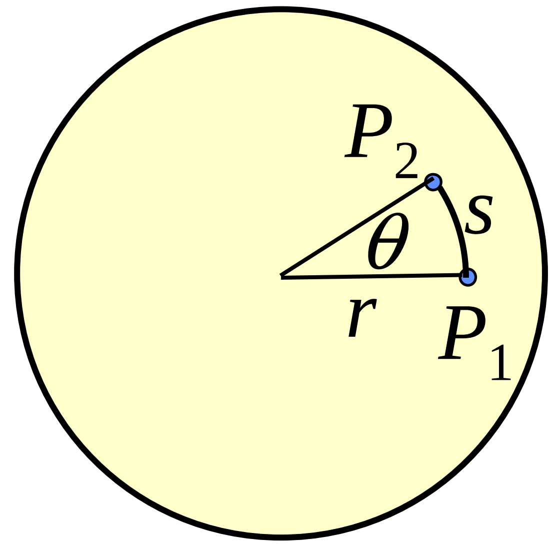
Fixed axis of rotation: A single-nonchanging axis around which the object rotates.

Rigid bodies: Objects that have a definite unchanging shape.

Translational motion: Motion with a fixed direction of net force.

Review of Terminolgy

$$\theta = s/r$$



$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$

$$v_{\mathrm{T}} = ds/dt = r\, d\theta/dt = r\omega$$

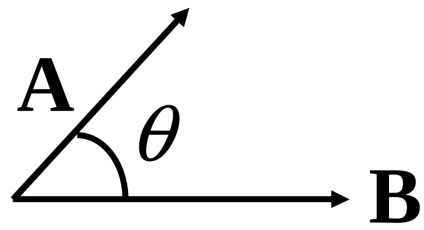
$$\left.\begin{array}{l} a_{\mathrm{T}} = dv_{\mathrm{T}}/dt = r\, d\omega/dt = r\alpha \\ a_C = v_{\mathrm{T}}^2/r = \omega^2 r \end{array}\right\} \quad \mathbf{a} = \mathbf{a}_{\mathrm{C}} + \mathbf{a}_{\mathrm{T}}$$

$$\omega = 2\pi\nu \qquad T = 1/\nu$$

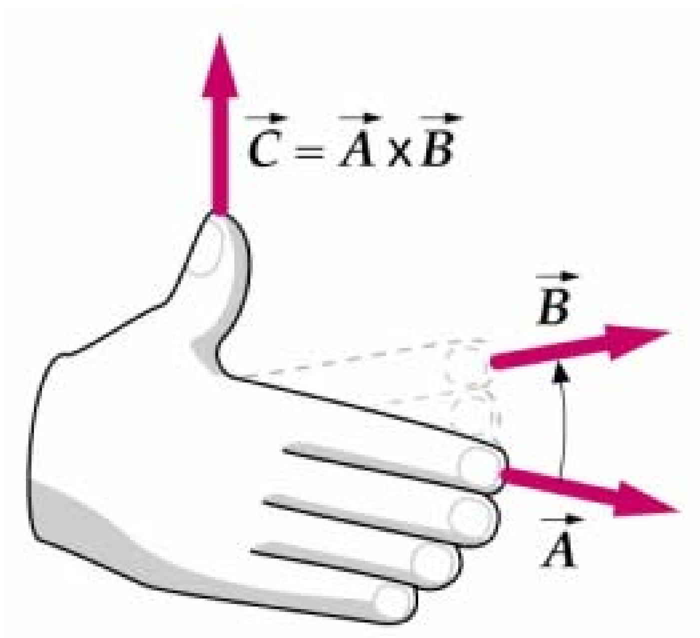
Vector (Cross) Product

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$C = AB \sin\theta$$



The direction of **C** is given by the right hand rule described in the text.



We can write the components of the cross product as:

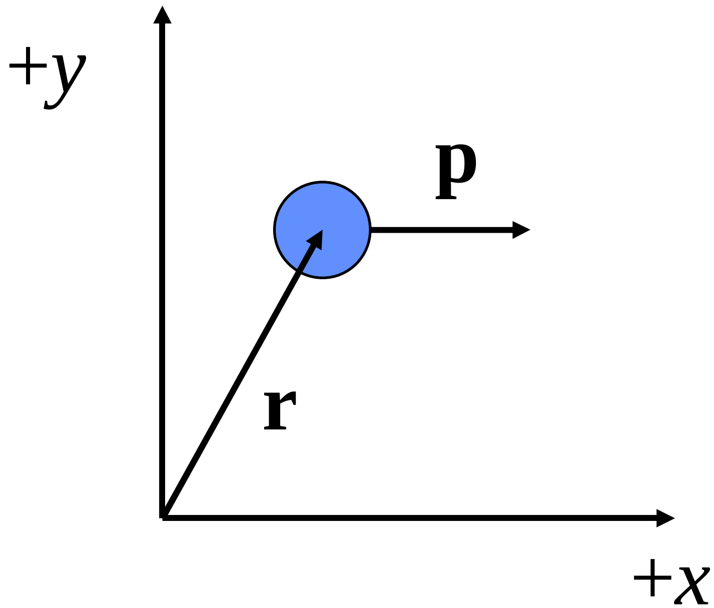
$$C_x = A_y B_z - A_z B_y$$
$$C_y = A_z B_x - A_x B_z$$
$$C_z = A_x B_y - A_y B_x$$

This can be easily remembered by using the determinant form:

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

(Orbital) Angular Momentum

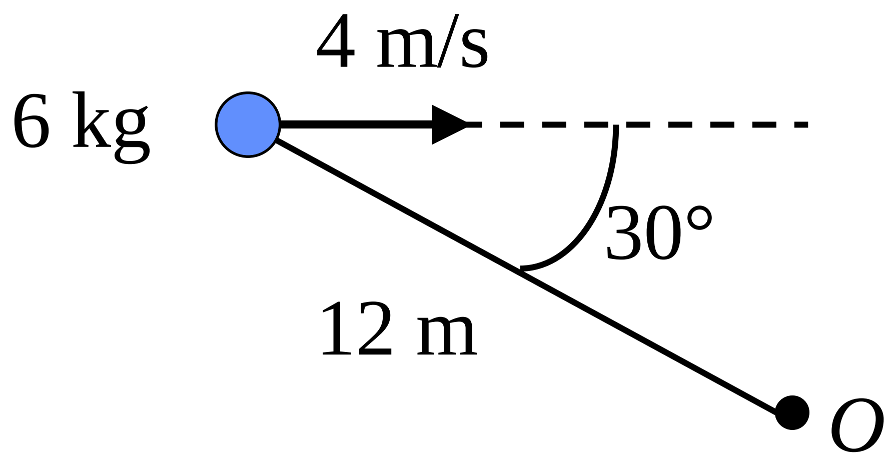


$$\mathbf{l} = \mathbf{r} \times \mathbf{p}$$

Linear momentum of a single particle: $\mathbf{p}$
Linear momentum of a system of particles: $\mathbf{P} = \sum \mathbf{p}$
Angular momentum of a single point particle: $\mathbf{l}$
Angular momentum of a system of particles: $\mathbf{L} = \sum \mathbf{l}$

Interactive Question

A 6-kg particle moves to the right at 4 m/s as shown. The magnitude of its angular momentum in kg·m^2/s about the point *O* is:

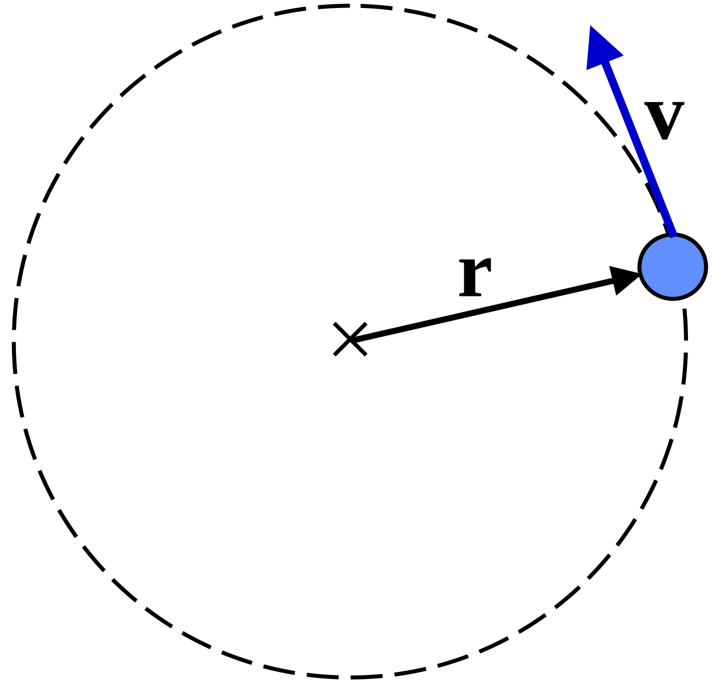


A) Zero
B) 24
C) 144
D) 288
E) 249

Problem: What is the angular momentum of a 4.0 kg object traveling with a velocity of $\mathbf{v} = 2.3\mathbf{i} + 1.5\mathbf{j}$ m/s when it is at (6.8, 5.6) m relative to the point (–3.0, –2.0)?

Angular Momentum for Circular Motion

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}$$
$$= \mathbf{r} \times m\mathbf{v}$$



The direction (in this case) for $\mathbf{l}$ is up, out of the paper. The magnitude is

$$l = rmv \sin\theta = rmv$$
$$= rm\,(\omega r) = mr^2\omega$$

And since $\boldsymbol{\omega}$ also points up out of the paper, $\mathbf{l} = mr^2\boldsymbol{\omega}$

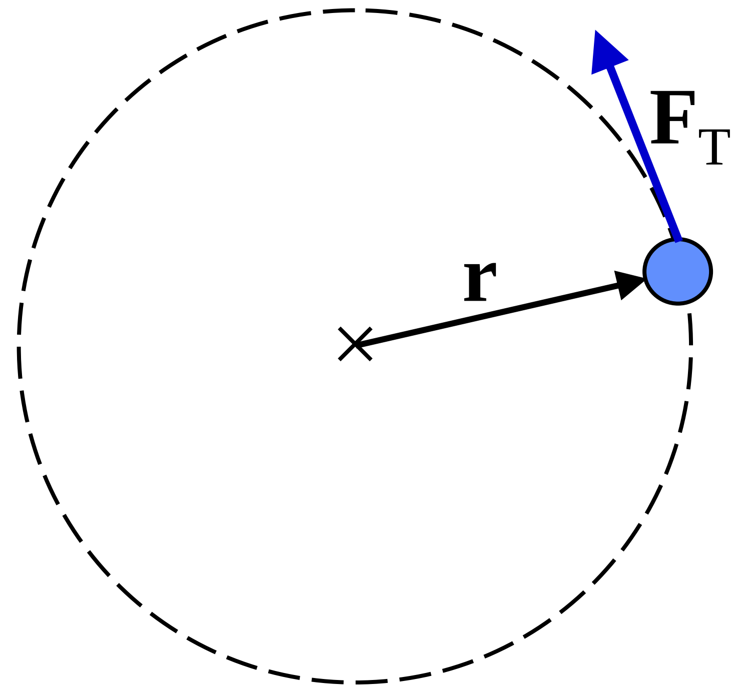
For this case, we define the moment of inertia $I = mr^2$

$$\mathbf{l} = I\boldsymbol{\omega}$$

Torque for Circular Motion

$\sum \mathbf{F} = m\mathbf{a}$

Consider the tangential motion of a rotating object with a tangential force acting on the object.



$F_\mathrm{T} = ma_\mathrm{T}$

$rF_\mathrm{T} = mr\, a_\mathrm{T} = mr\, r\alpha$

$rF_\mathrm{T} = mr^2\, \alpha$

$\tau = mr^2\, \alpha = I\alpha$

τ is the torque

Since the angular acceleration has a direction out of the page,

$$\tau = I\alpha$$

If there is more than one force with a tangential component,

$$\sum\tau = \sum(mr^2)\alpha$$
$$\sum\tau_{\text{ext}} + \sum\tau_{\text{int}} = \sum(mr^2)\alpha$$
$$\sum\tau_{\text{ext}} = \sum(mr^2)\alpha$$
$$\sum\tau = I\,\alpha$$

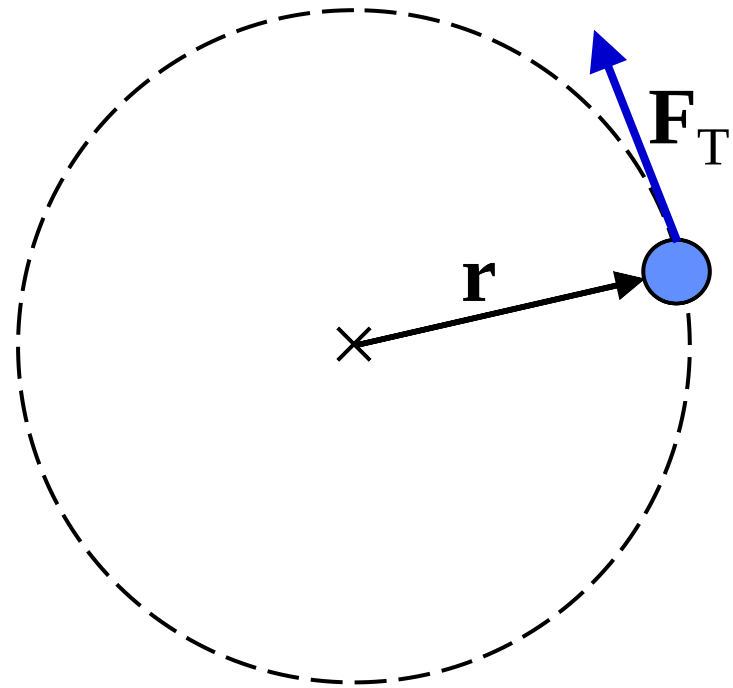
where $I = \sum(mr^2)$ is the general form of the rotational inertia, or moment of inertia.

This is one form of Newton's second law for rotation.

The moment of inertia depends on the object and the axis of rotation. More on this later.

General Definition of Torque

In the previous example we defined $\tau = rF_{\mathrm{T}}$ and defined the direction of the torque as up out of the page.



It makes sense, then, that the general definition of torque is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Consider now a system of particles:

$$\mathbf{L} = \sum_i \mathbf{l}_i = \sum_i (\mathbf{r}_i \times \mathbf{p}_i)$$

$$d\mathbf{L}/dt = \sum_i \{(d\mathbf{r}_i/dt \times m\mathbf{v}_i) + (\mathbf{r}_i \times m\, d\mathbf{v}_i/dt)\}$$

$$= \sum_i \{(\mathbf{v}_i \times m\mathbf{v}_i) + (\mathbf{r}_i \times m\mathbf{a}_i)\}$$

$$= \sum_i \{0 + (\mathbf{r}_i \times m\mathbf{a}_i)\}$$

$$= \sum_i (\mathbf{r}_i \times \sum_j \mathbf{F}_j) = \sum_i \sum_j \boldsymbol{\tau}_j$$

$$= \sum \boldsymbol{\tau}_{net} = \sum \boldsymbol{\tau}_{ext} + \sum \boldsymbol{\tau}_{int}$$

$$d\mathbf{L}/dt = \sum \boldsymbol{\tau}_{ext}$$

$$\sum \boldsymbol{\tau}_{ext} = d\mathbf{L}/dt$$

This is the more general form of Newton's 2nd law for rotation.

Summary

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

General Case

$\sum \boldsymbol{\tau}_{ext} = d\mathbf{L}/dt$

$\mathbf{l} = \mathbf{r} \times \mathbf{p}$

$I = \sum(mr^2)$

Specific Case

If the moment of inertia doesn't change:

$\sum \boldsymbol{\tau}_{ext} = I\alpha$

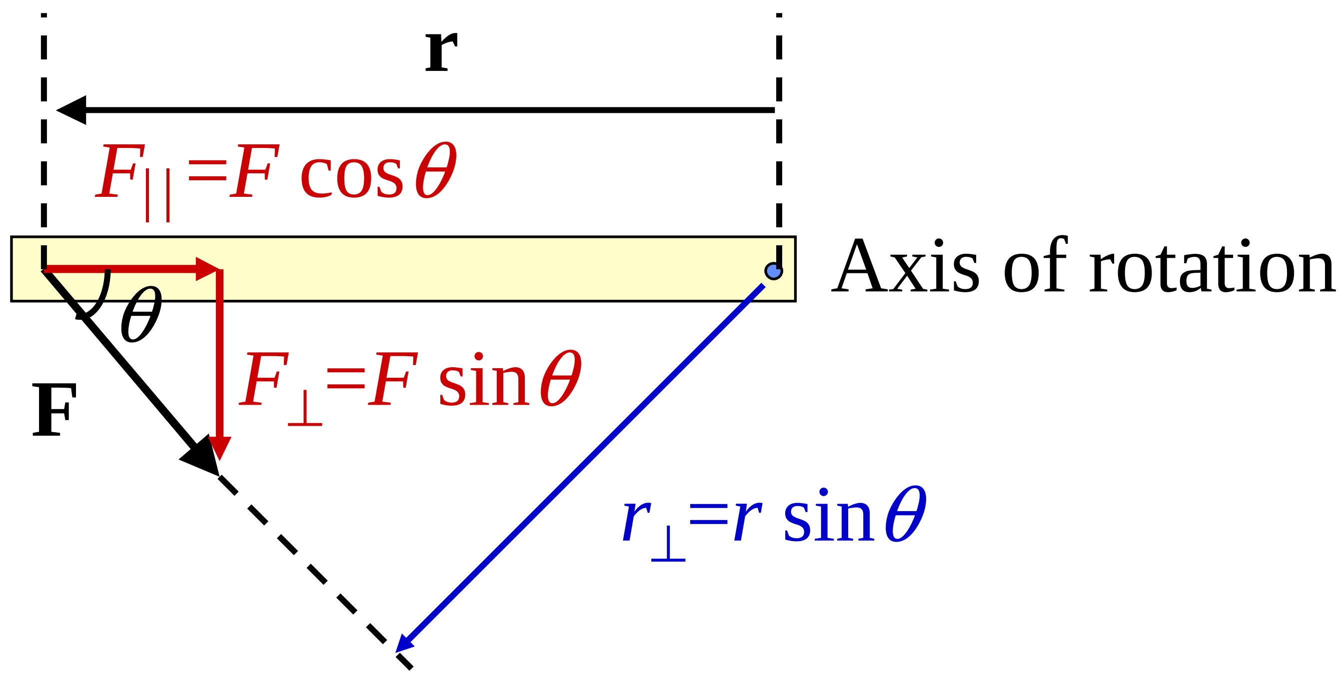
For a "point" object in a circular orbit:

$\mathbf{l} = I\omega$

$I = mr^2$

Now let's do some problems with all of these concepts

Torque



$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

The direction of the torque is up out of the page and the magnitude is

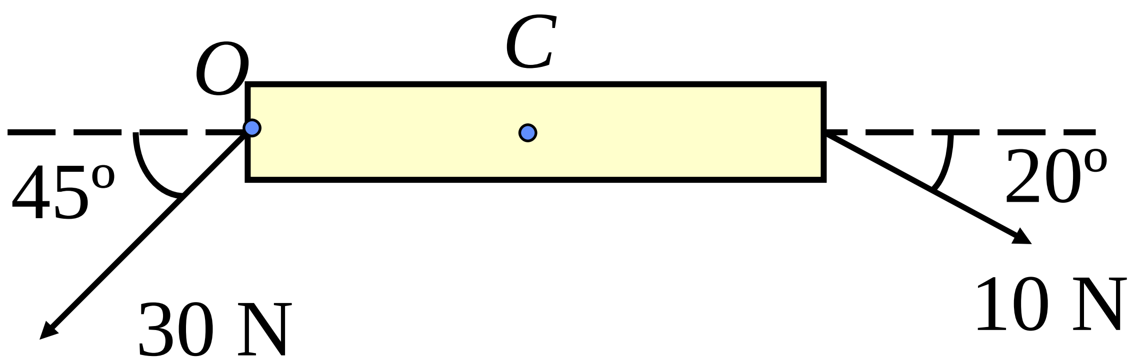
$\tau = rF\sin\theta = Fr_{\perp} = rF_{\perp}$

$r_{\perp}$ is often called the "lever arm"

Problem: Consider the object shown below, and the forces acting on the object. Calculate the torque around an axis perpendicular to the paper through

(a) point *O*

(b) point *C*.



Problem: A penny with a mass of 2.5 grams sits on a turntable a distance of 6.6 cm from the center. The turntable starts from rest and spins up to a maximum angular speed of 33 rev/min in 2.2 seconds rotating counterclockwise. The penny does not slide with respect to the turntable.

a) Assuming a constant torque what is the torque on the penny with respect to the center of the turntable?
b) What is the angular speed of the penny after 1.1 seconds with respect to the center of the turntable?
c) What is the angular momentum of the penny after 1.1 seconds with respect to the center of the turntable?
d) What is the magnitude of the total force of friction acting on the penny after 1.1 seconds?

Conservation of Angular Momentum

$$\sum \boldsymbol{\tau}_{\text{ext}} = d\mathbf{L}/dt$$

When there is no net external torque, then the angular momentum does not change with time. It is conserved.

$$d\mathbf{L}/dt = 0$$

$$\mathbf{L}_{\text{final}} = \mathbf{L}_{\text{initial}}$$

There may be no external torque for many reasons, including the case when the line of force for all forces passes through the axis of rotation. ($\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$)

Problem: Show how Kepler's second law, that the area swept out by a planet moving in an ellipse is always the same for equal time intervals, is a result of the conservation of angular

Moment of Inertia

For a single "point" object with mass m, rotating at a distance of r around the axis of rotation, $I = mr^2$

To calculate the rotational inertia for a more complicated object, we simply sum up the rotational inertia of each part of the object.

Problem: Suppose a baton is 1.0 m long with weights on each end weighing 0.3 kg. Neglect the mass of the bar and consider the weights as point objects. What is the moment of inertia for a baton

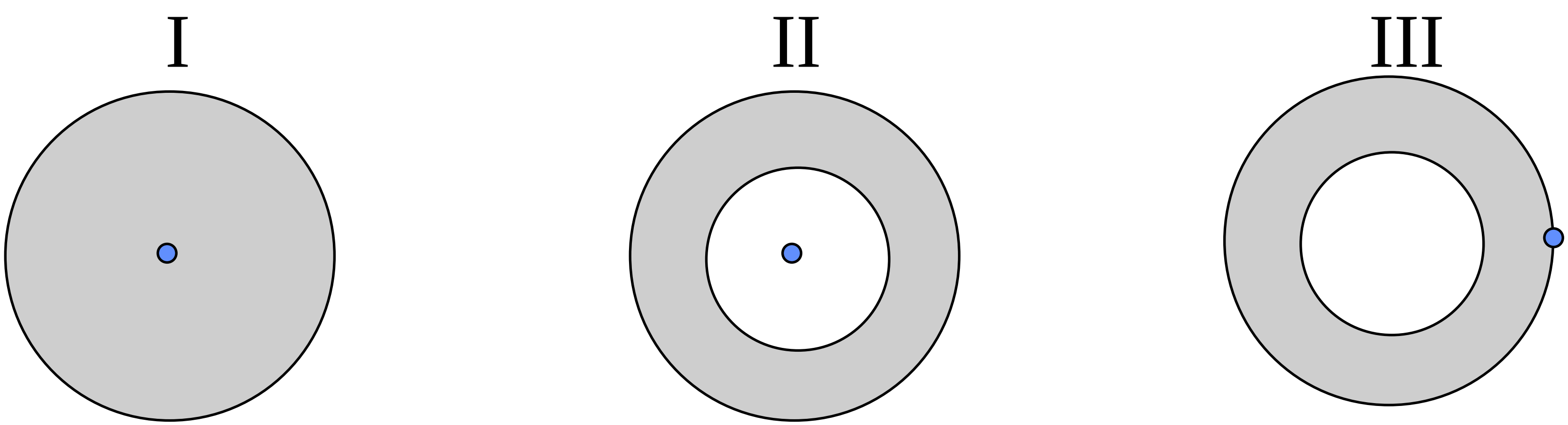
(a) spinning around its center?

(b) spinning around one end of the baton?

The rotational inertia depends on both the axis of rotation, and how the mass is distributed.

Interactive Question

Three thin disks with uniform mass density have the same mass, and radius. Rank the objects in order of increasing moment of inertia. The axis of rotation is shown as the dark dot and is perpendicular to the paper.



A) I, II, III
B) III, II, I
C) II, III, I
D) II, I, III
E) More information is needed

Calculating the Moment of Inertia

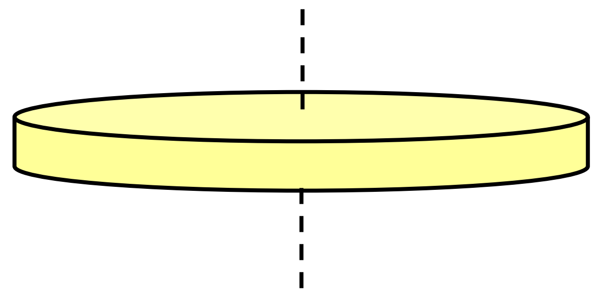
For an object with a continuous mass distribution, the sum becomes an integral and we get

$$I = \int r^2 \, dm$$

Moments of inertia that have been calculated this way are often found in a table. (See Table 10.1)

You don't have to memorize these formulas. They will be given (or I will ask you to derive them.)

<u>Problem:</u> What is the moment of inertia of a thin disk with a mass m, radius R, and thickness t, rotating through its center of mass along an axis perpendicular to the plane of the disk?



The Parallel Axis Theorem

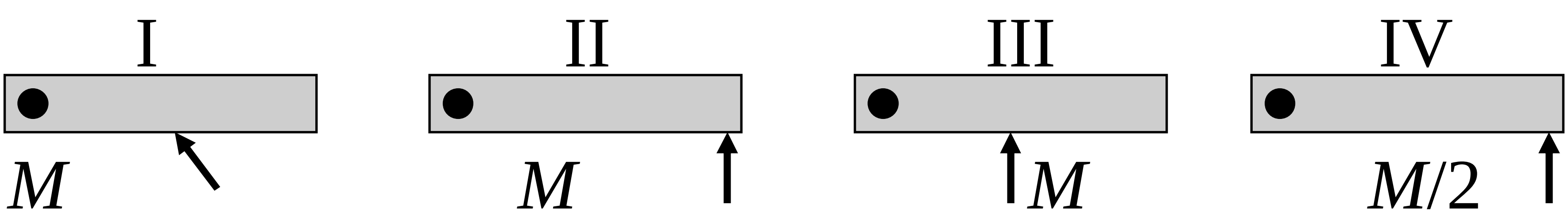
If you know the moment of inertia about any axis that passes through the center of mass of an object, you can find its moment of inertia about any other axis parallel to that axis with the parallel axis theorem which states

$$I = I_{\mathrm{CM}} + Mh^2$$

where I_{CM} is the moment of inertia about the center of mass, M is the mass of the object, and h is the perpendicular distance between the two axes. The proof of this theorem is given in the book.

Interactive Question

Four uniform long rods with the same length but different masses are pushed with the same force as shown. Rank the cases in the order of increasing angular acceleration. The axis of rotation is perpendicular to the paper and shown by the black dot.



A) I, II, III IV

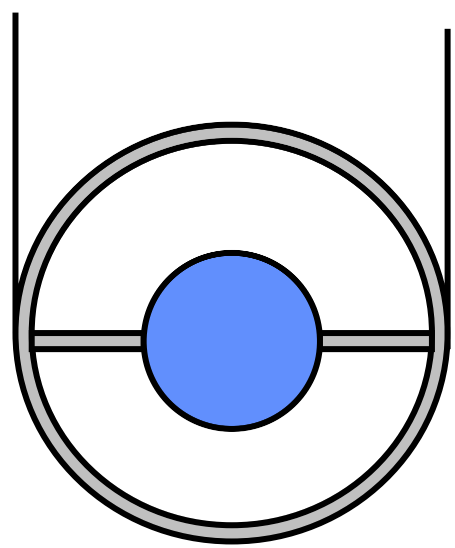
B) III, I, II, IV

C) I, III, IV, II

D) I, III, II, IV

E) None of the above

Problem: A model of Uranus (with the rings held on by thin rods) hangs from two wires as shown. What is the moment of inertia about the wire? The radius of Uranus is R=0.25 m and its mass is M=1.0 kg. The ring is a thin hoop with radius of r=0.5 m and a mass of m=0.25 kg. The rods each have negligible mass. If one of the wires breaks, what is the moment of inertia of the model as it rotates around the other wire?

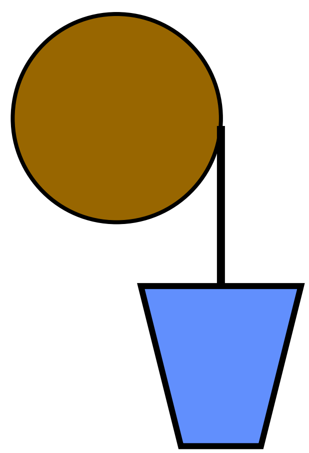


Problem: A cylindrical pulley with a mass of M=3.00 kg and a radius of R=0.400 m is used to lower a bucket with a mass of m=2.00 kg into a well. The bucket starts from rest and falls for 3.00 s.

(a) What is the linear acceleration of the falling bucket?

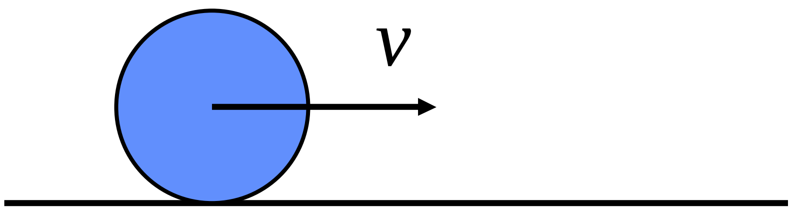
(b) How far does it drop?

(c) What is the angular acceleration of the cylinder?



Introduction to Rolling Motion

Consider the motion of an object that is rolling without slipping.



At point of contact, $v_c = 0$

At top of wheel, $v_{top} = 2v$

If the rotational velocity is given by ω and the radius of the rolling object is r, then it is clear that $v = \omega r$.

Rotational Kinetic Energy

An object spinning with angular velocity ω has a rotational kinetic energy given by

$$K = (1/2)I\omega^2$$

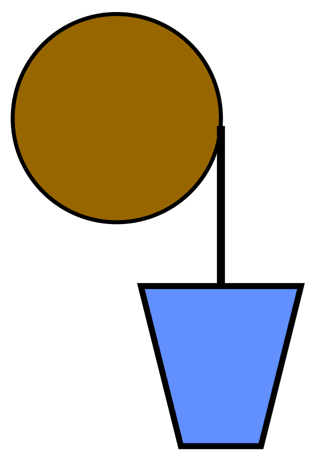
exactly analogous to translations kinetic energy.

An object that is rolling with a center of mass velocity v, and an angular velocity ω about its center of mass has a total kinetic energy given by:

$$K = (1/2)mv^2 + (1/2)I_{CM}\omega^2$$

Of course, when calculating gravitational potential energy, the mass of an object can be considered as entirely located at the center of mass.

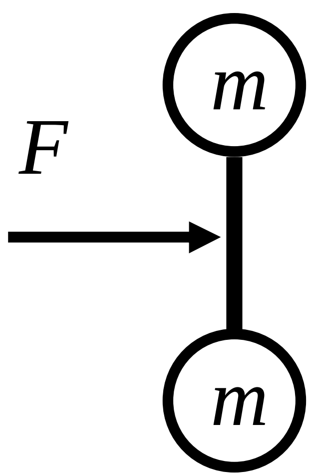
Problem: A cylindrical pulley with a mass of M=3.00 kg and a radius of R=0.400 m is used to lower a bucket with a mass of m=2.00 kg into a well. The bucket starts from rest and falls for a distance of 25.2 m. Use conservation of energy to find the speed of the bucket after it has fallen this distance. Neglect any dissipative forces.



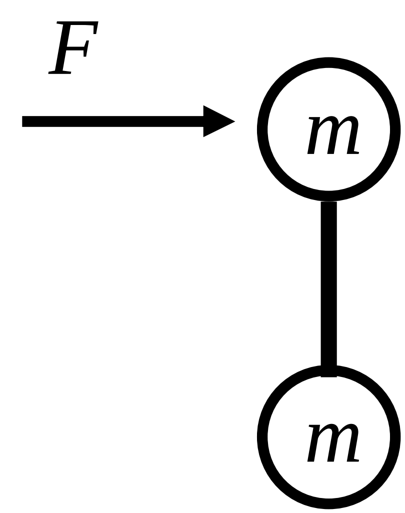
Interactive Question

A force F is applied to a dumbbell for a time interval Δt, first as in (a) and then as in (b). In which case does the dumbbell acquire the greater center-of-mass speed?

(a)



(b)



A) (a)

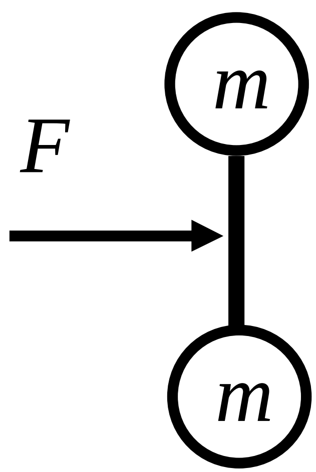
B) (b)

C) no difference

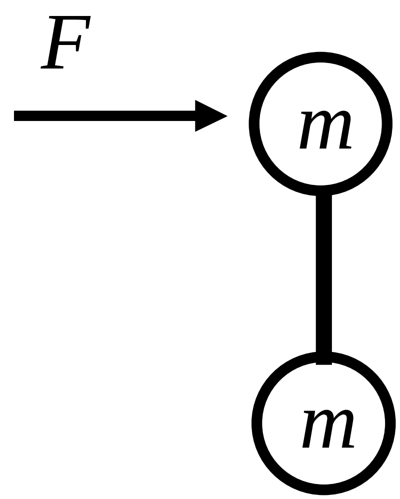
Interactive Question

A force F is applied to a dumbbell for a time interval Δt, first as in (a) and then as in (b). In which case does the dumbbell acquire the greater energy?

(a)



(b)



A) (a)

B) (b)

C) (no difference

Interactive Question

A hollow cylinder of mass M and radius R rolls down an inclined plane. A block of mass M slides down an identical inclined plane. If both objects are released at the same time

A) the block will reach the bottom first.
B) the cylinder will reach the bottom first.
C) the block will reach the bottom with greater kinetic energy
D) the cylinder will reach the bottom with greater kinetic energy
E) both the block and the cylinder will reach the bottom at the same time.

Interactive Question

A solid sphere (S), a thin hoop (H), and a solid disk (D), all with the same radius, are allowed to roll down an inclined plane without slipping. In which order will they arrive at the bottom? (The fist one down listed first).

A) H,D,S

B) H,S,D

C) S,D,H

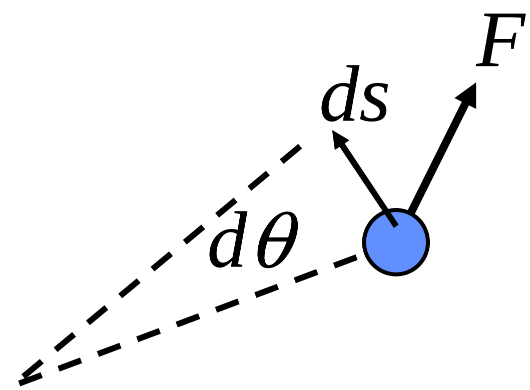
D) S,H,D

E) D,H,S

Problem: Two bicycles roll down a hill which is 20 m high. Both bicycles have a total mass of 12 kg and 700 mm diameter wheels (r = 0.350 m). The first bicycle has wheels with a mass of 0.60 kg each, and the second bicycle has wheels with a mass of 0.30 kg each. Neglecting air resistance, which bicycle has the faster speed at the bottom of the hill? (Consider the wheels to be thin hoops).

Rotational Work-Energy Theorem

Consider a force acting on an object so that it rotates a angle $d\theta$.



$$W = \int \mathbf{F} \cdot d\mathbf{s} = \int F_T R \, d\theta = \int \tau \, d\theta$$

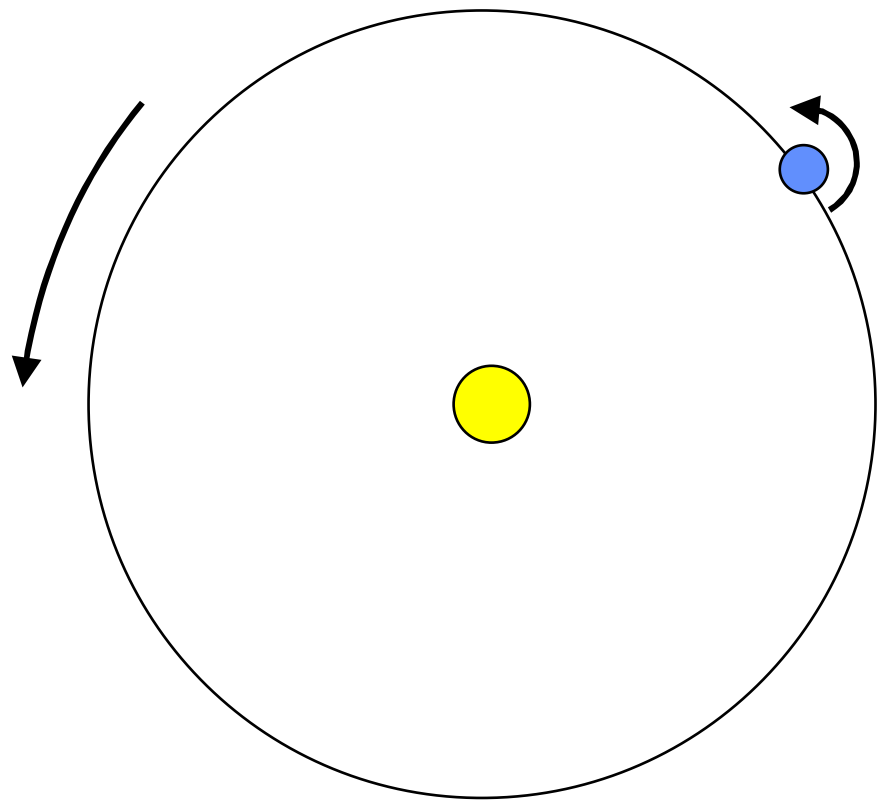
where ϕ is the angle between the applied force and the tangential motion, and θ is the angle between the applied force and the radial vector from the axis of rotation to the force so that $\phi + \theta = 90°$.

In a similar manner: $P = dW/dt = \tau \, d\theta/dt = \tau\omega$

Problem:

a) What is the kinetic energy of the earth's rotation and the kinetic energy of the earth's orbit around the sun? What is the total kinetic energy of the earth?

b) What is the angular momentum of the earth's rotation and of the earth's orbit around the sun? What is the total angular momentum of the earth?



$m_E = 5.97 \times 10^{24}$ kg
$r_E = 6.38 \times 10^{6}$ m
$r_S = 1.49 \times 10^{11}$ m

More on Rolling Motion

Rolling motion can be thought of as the translational motion of the center of mass plus the rotational motion around the center of mass.

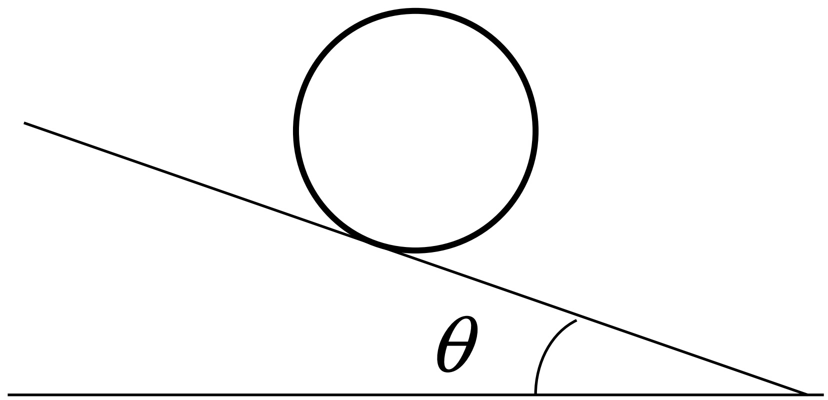
Or it can be thought of as purely rotational motion about the point in contact with the ground

Either way gives the same answer for the total kinetic energy.

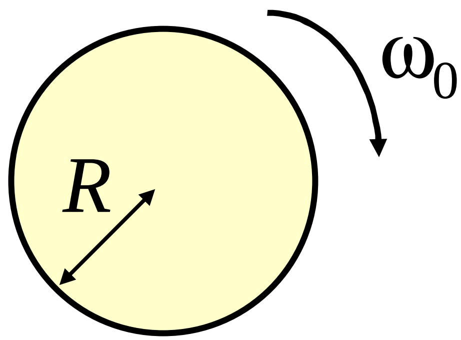
$$K = (1/2)I\omega^2 = (1/2)(mR^2/2 + mR^2)\omega^2 = (3/4)mR^2\omega^2$$

$$K = (1/2)mv^2 + (1/2)I\omega^2 = (1/2)mv^2 + (1/2)(mR^2/2)\omega^2$$
$$= (1/2)mR^2\omega^2 + (1/2)(mR^2/2)\omega^2 = (3/4)mR^2\omega^2$$

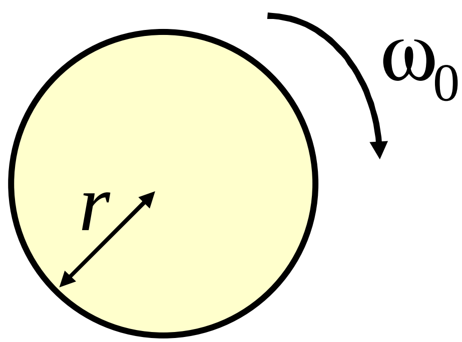
Problem: A solid sphere rolls down an incline without slipping. If the acceleration of the center of mass is $0.2g$, what angle is the incline?



Problem: A uniform solid sphere is set rotating about a horizontal axis with an angular speed ω_0 and is placed on the floor. The coefficient of kinetic friction between the floor and the sphere is μ. What is the speed of the center of mass of the ball when it starts rolling without slipping.



Problem: A cylindrical rod is set rotating with an angular speed of ω_0, then placed on the floor. If the coefficient of kinetic friction between the rod and the floor is μ_k, what is the speed of the center of mass when the rod begins to roll without slipping?



More on Conservation of Angular Momentum

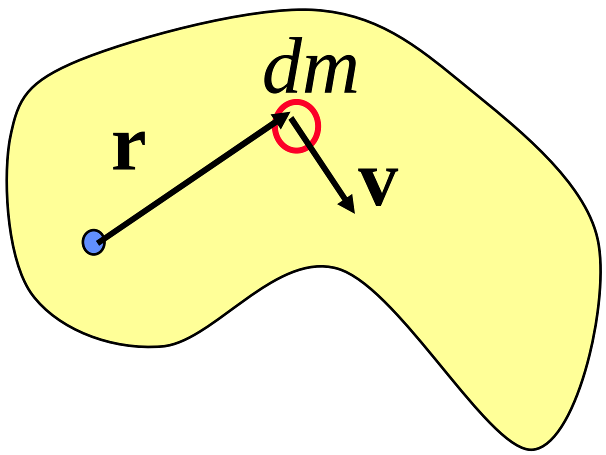
$$\sum \boldsymbol{\tau}_{\text{ext}} = d\mathbf{L}/dt$$

So if there is no external torque, $d\mathbf{L}/dt = 0$, angular momentum is conserved. When will there be no external torque?

We know for a "point" particle that $\mathbf{l} = \mathbf{r} \times \mathbf{p}$, and that if the particle is in a circular orbit, this becomes $\mathbf{l} = I\omega$. But what is L for a rigid object rotating around a fixed axis of rotation?

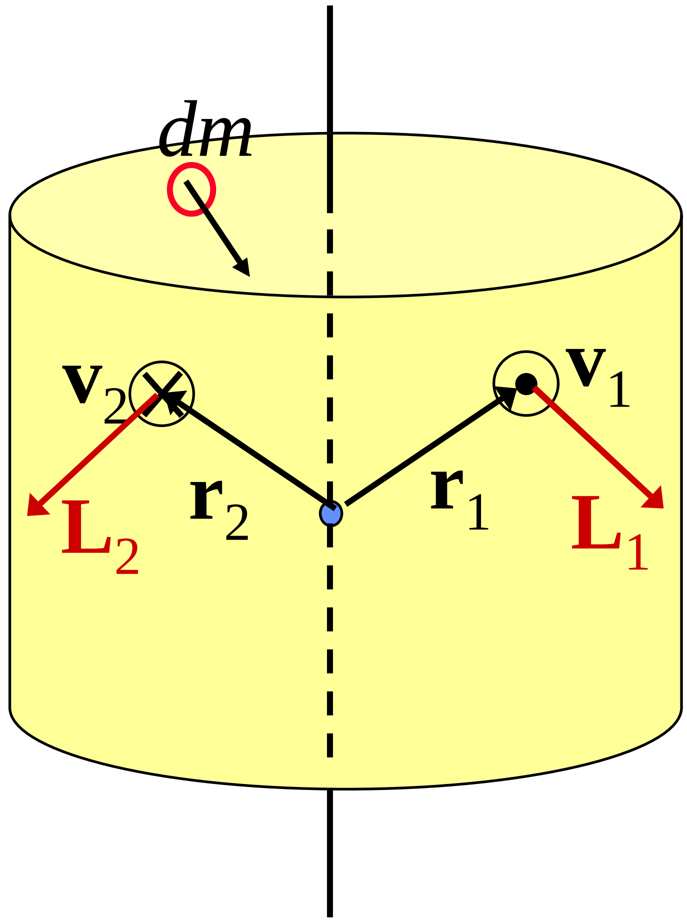
Let's show that $\mathbf{L} = \sum_i \mathbf{l}_i = \sum_i (\mathbf{r}_i \times \mathbf{p}_i) = I\boldsymbol{\omega}$ for any rigid body rotating around a fixed axis of rotation.

First, consider the angular momentum for a rigid body rotating around a fixed axis of rotation in a plane, so that the angular momentum is in a fixed direction along the axis of rotation.



$$L = \int rv \sin\theta \, dm = \int r \, \omega r \sin(90^{o}) \, dm$$
$$= \int r^2 \omega \, dm = \omega \int r^2 \, dm$$
$$L = I\omega$$

Now consider two points on an object that is symmetric about its axis of rotation.



The horizontal components of $\mathbf{L}_1$ and $\mathbf{L}_2$ cancel and only the vertical components contribute to the total angular momentum. So just like a single plane rotating, $L = I\omega$.

Interactive Question

An ice skater performs a pirouette by pulling her outstretched arms close to her body. What happens to her moment of inertia about the axis of rotation?

A) It does not change.
B) It increases.
C) It decreases.
D) It changes, but it is impossible to tell which way.

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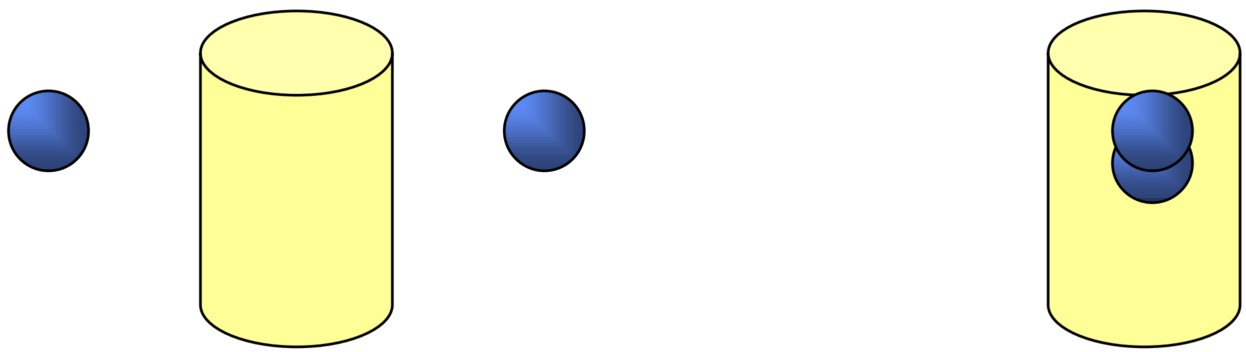
C) It decreases.
D) It changes, but it is impossible to tell which way.

Problem: A student is sitting on a swivel seat and is holding a 2.0 kg weight in each hand. If he is rotating at 1 rev/s (6.28 rad/s) when the weights are held in outstretched arms .75 m from the axis of rotation, how fast is he rotating when he pulls the weights in to the axis of rotation? (The rest of his body can be approximated as a cylinder with mass of 72 kg and radius of .25 m).



Interactive Question

A ball on a string is rotating in a circle. The string is shortened by pulling it through the axis of rotation. What happens to the angular velocity and the tangential velocity of the ball?

angular velocity	tangential velocity
A) increases	decreases
B) increases	stays the same
C) increases	increases
D) stays the same	stays the same
E) stays the same	increases

Interactive Question

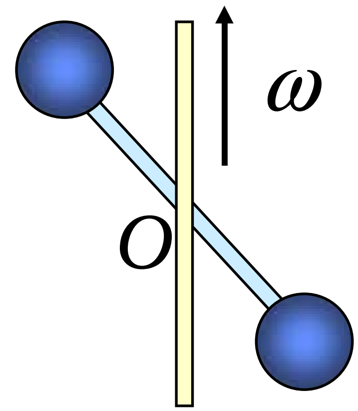
An ice skater performs a pirouette by pulling her outstretched arms close to her body. What happens to her rotational kinetic energy about the axis of rotation?

A) It does not change.
B) It increases.
C) It decreases.
D) It changes, but it is impossible to tell which way.

Problem: A child of mass 25 kg runs with a speed of 2.5 m/s and jumps on a merry-go-round along a path tangential to the rim. The merry-go-round has a moment of inertia of 500 $kg \cdot m^2$ and a radius of 2 m. What is the final angular velocity of the merry-go-round and child?

Interactive Question

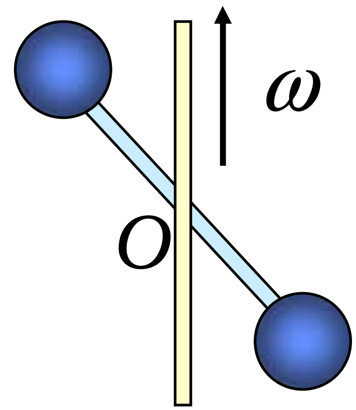
A baton with two spheres at its end is rotated around an axis as shown. As the baton spins which of the following is true.

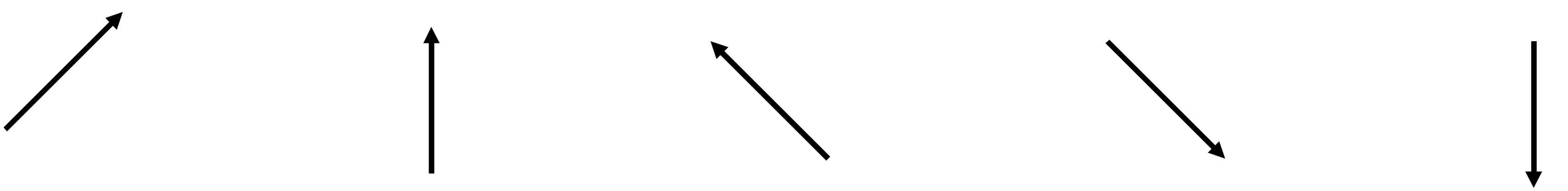


A) The angular momentum of the baton doesn't change
B) The angular velocity points in the same direction as the angular momentum.
C) The angular momentum changes, but there is no torque applied to the baton.
D) A net external torque causes the angular momentum to change
E) More than one of the above is true

Interactive Question

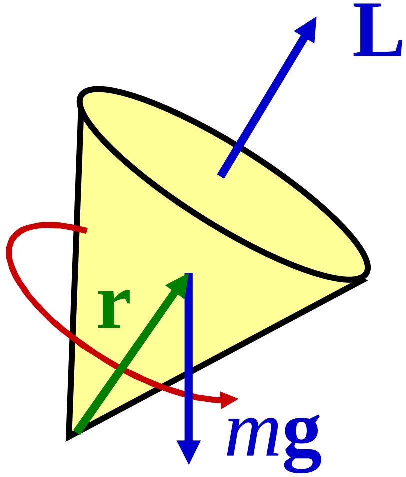
A baton with two spheres at its end is rotated around an axis as shown. Neglect the mass of the rod connecting the spheres. Which direction is the angular momentum pointing when the baton is in the position shown?



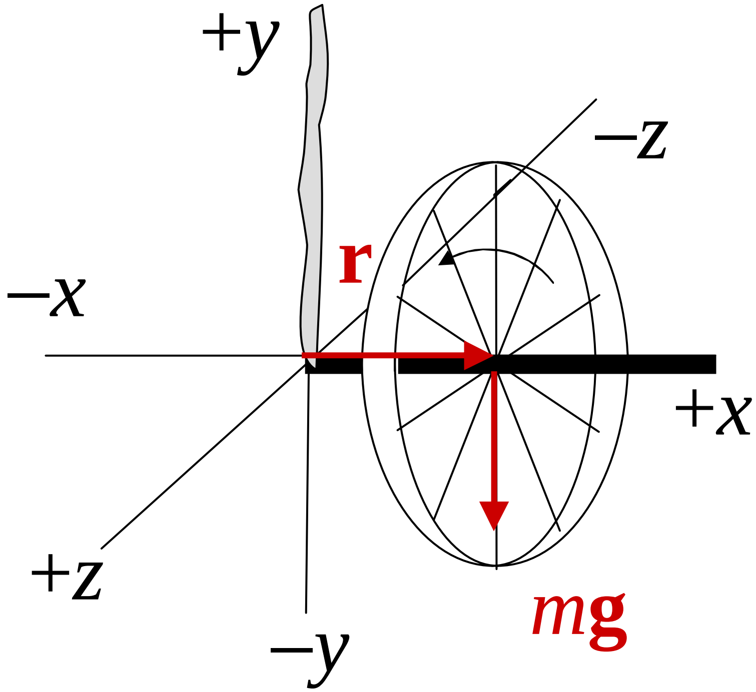


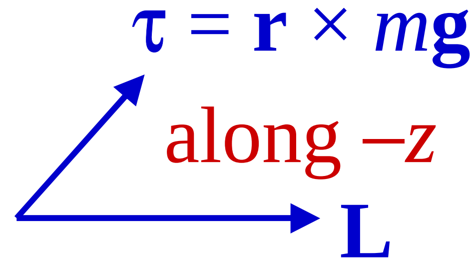
(A) (B) (C) (D) (E)

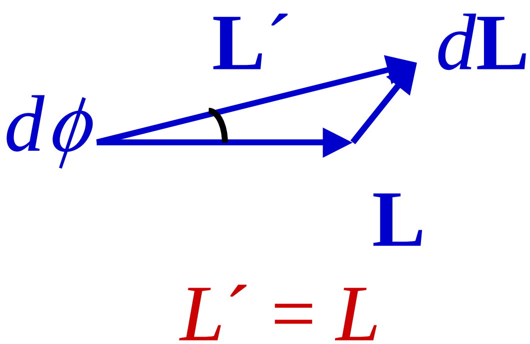
Precession of a Top



$$\otimes\ \boldsymbol{\tau} = \mathbf{r} \times m\mathbf{g} = d\mathbf{L}/dt$$







$$d\phi = dL/L = \tau\, dt/L = mgr\; dt/L$$

$$\omega_{\mathrm{p}} = d\phi/dt = mgr/L = mgr/I_{\mathrm{w}}\omega_{\mathrm{w}}$$

Static Equilibrium

The conditions for static equilibrium in two dimensions:

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum \tau = 0$$

The net torque must be zero around *any* axis of rotation.

<u>Problem:</u> A board with uniform mass density and a weight of 40.0 N supports two children weighing 500 N and 350 N. The support is placed under the center of gravity of the board, and the 500 N child is 1.50 m from the center.

(a) What is the force which the support exerts?

(b) Where should the 350 N child sit to balance the board?

Interactive Question

A heavy boy and a lightweight girl are balanced on a massless seesaw. If they both move forward so that they are one-half their original distance from the pivot point, what will happen to the seesaw?

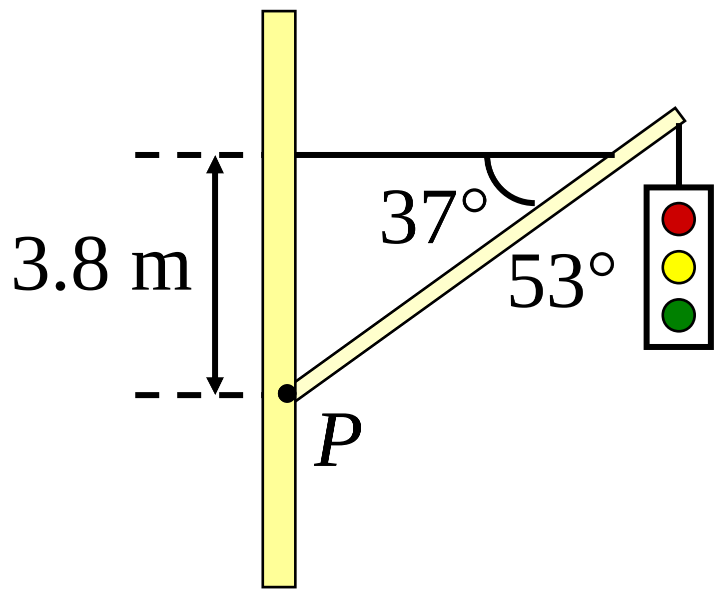
A) The side the boy is sitting on will tilt downward.

B) The side the girl is sitting on will tilt downward.

C) Nothing, the seesaw will still be balanced.

D) It is impossible to say without knowing the masses and the distances.

Problem: A traffic light hangs from the end of a long pole as show. The pole has a length L = 7.5 m long and a mass of 8.0 kg. The mass of the light is 11.0 kg. Determine the tension in the horizontal cable and the vertical and horizontal components of the force on the pivot point P?



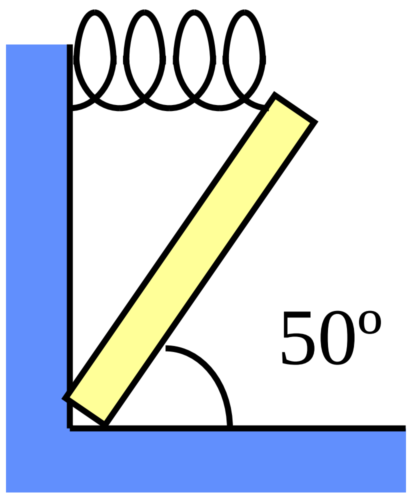
L = 7.5 m

$r_{\perp}$ = 3.8 m

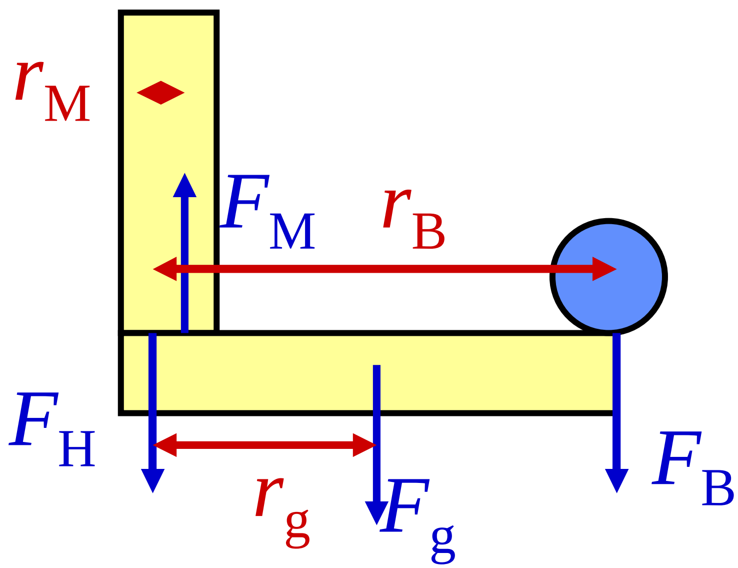
M = 8.0 kg

m = 11.0 kg

Problem: A 10.1 kg uniform board is wedged into a corner and held by a spring attached to its end at a 50.0° angle with respect to the horizontal direction. The spring has a spring constant of 176 N/m. By how much does the spring stretch?



Problem: You hold your forearm out horizontally and hold a 50 N object in your hand (located 0.35 m from the elbow joint). Your bicep muscle is attached at a distance of 0.030 m from the elbow joint. The mass of your forearm is 1.3 kg with the center-of-mass of the forearm located 0.17 m from the elbow joint. What is the force of the humerous (the bone between the shoulder and the forarm) and the bicep on the arm?



$F_B = 50$ N

$F_g = (1.3 \text{ kg})(9.8 \text{ m/s}^2)$

$r_B = 0.35$ m

$r_g = 0.17$ m

$r_M = 0.030$ m

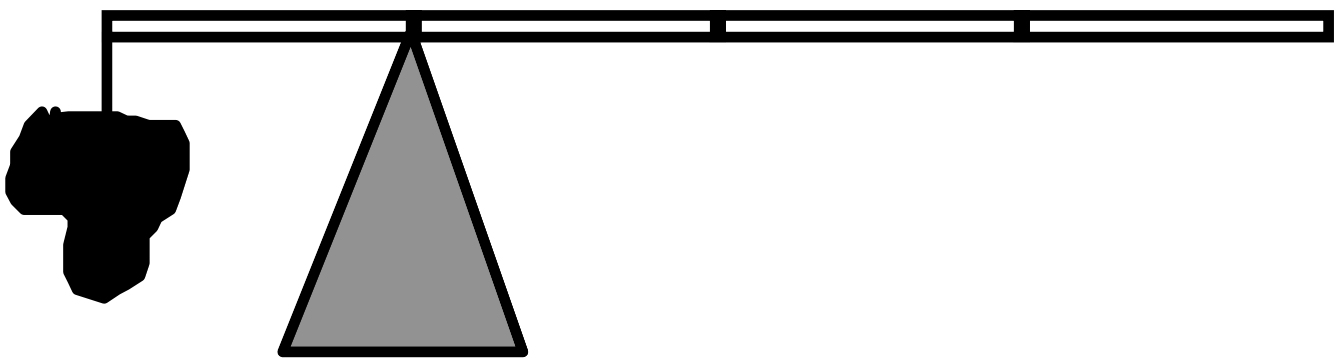
<u>Problem:</u> The large quadriceps muscle in the upper leg terminates at its lower end in a tendon attached to the upper end of the tibia. The forces on the lower leg are modeled in the figure on the next page where **T** is the tension in the tendon, **C** is the weight of the lower leg, **F** is the weight of the foot, and **B** is the force of the femur on the tibia. Assume $C = 30.0$ N, $F = 12.5$ N, and the leg is in the position shown in the figure. The tendon is attached one fifth of the way down the lower leg, and the center of mass of the lower leg is at its geometric center.

a) Find the tension of the tendon, **T.**

b) Find the *x* and *y* components of the force of the femur on the tibia, **B**.

Interactive Question

A 1-kg rock is suspended by a massless string from one end of a 1-m measuring stick. What is the weight of the measuring stick if it is balanced by a support force at the 0.25-m mark?



A) 0.25 kg B) 0.5 kg C) 1kg

D) 2 kg E) 3kg