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The renormalization group and spontaneous compactification of a higher-dimensional scalar field theory in curved spacetime

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Abstract

The renormalization group (RG) is used to study the asymptotically free ϕ_6^3-theory in curved spacetime. Several forms of the RG equations for the effective potential are formulated. By solving these equations we obtain the one-loop effective potential as well as its explicit forms in the case of strong gravitational fields and strong scalar fields. Using zeta function techniques, the one-loop and corresponding RG improved vacuum energies are found for the Kaluza-Klein backgrounds $R^4 \times S^1 \times S^1$ and $R^4 \times S^2$. They are given in terms of exponentially convergent series, appropriate for numerical calculations. A study of these vacuum energies as a function of compactification lengths and other couplings shows that spontaneous compactification can be qualitatively different when the RG improved energy is used.

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I. Introduction.

The renormalization group (RG) has long been used to "improve" loop corrections in perturbative quantum field theory. Gell-Mann and Low [1] first used it to study the asymptotic behavior of Green's functions and in the classic work of Coleman and Weinberg [2] the RG equation was used to improve the effective potential and study spontaneous symmetry breaking. This is just one of the many different applications that the RG has had in quantum filed theory. Recently it has been employed to put lower limits on the Higg's mass of the standard model [3]. In this paper we put it to use in a renormalizable Kaluza-Klein model, arguing that RG improvments are necessary if stability of the internal dimensions are to be correctly predicted. We will also develop the RG technique in order to study the scalar effective potential in this model.

We start in Sec. II using the RG equations to arrive at the 1-loop effective potential starting from a classical ϕ^3 scalar field theory defined on a 6-dimensional curved space (denoted by ϕ^3_6). This is a renormalizable theory which is coupled to the curvature tensor and its square. We additionally use the RG equations to find the asymptotic behavior of the effective potential when either the gravitational field is strong or when the scalar field is intense. In Sec. III we give the 1-loop vacuum energies for this scalar field on backgrounds $R^4 \times S^1 \times S^1$ and $R^4 \times S^2$. We then compute RG improvements to these energies. We conclude that qualitative changes have occurred, i.e., minima have disappeared from the vacuum energy and that Kaluza-Klein stability will be correspondingly affected. In the conclusions we also mention other possible applications of the RG techniques in the context of Kaluza-Klein theories, as higher-derivative theories and renormalizable theories in the "modern sense."

II. A renormalizable self-interacting scalar theory in D=6 curved spacetime.

As an example of a renormalizable theory in higher-dimensional curved spacetime, we consider the following action in $D = 6$:

$$\begin{aligned}
L &= L_m + L_{ext}, \\
L_m &= -\frac{1}{2}\phi\Box\phi + \frac{1}{2}M^2\phi^2 + \frac{1}{3!}g\phi^3 + h\phi + \frac{1}{2}\xi R\phi^2 + \eta_1 R\phi \\
&\quad +\eta_3 R^2\phi + \eta_4 R_{\mu\nu}R^{\mu\nu}\phi + \eta_5 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\phi, \qquad (1)\\
L_{ext} &= -\Big(\Lambda + kR + \alpha_1 R^2_{\mu\nu\alpha\beta} + \alpha_2 R^2_{\mu\nu} + \alpha_3 R^2 + \alpha_4 R^3 + \alpha_5 RR^2_{\mu\nu} + \alpha_6 RR^2_{\mu\nu\alpha\beta} \\
&\quad + \alpha_7 R_{\mu\nu}R^{\mu}{}_{\sigma}R^{\nu\sigma} + \alpha_8 R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} + \alpha_9 R_{\mu\nu}R^{\mu\lambda\rho\sigma}R^{\nu}{}_{\lambda\rho\sigma} + \alpha_{10} R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\tau}R^{\rho\sigma\lambda\tau}\Big).
\end{aligned}$$

Here L_m and L_{ext} are the Lagrangians of matter and external fields, respectively, and ϕ is a scalar. The Lagrangian (1) represents the generalization to curved space of a renormalizable ϕ^3_6 theory [4]. Such a theory in curved spacetime was considered a few years ago in Refs. [5]-[7]. Here the notation of Ref. [6] will be adopted. In that reference, a one-loop analysis was carried out. The form of L_{ext} in (1), as well as of the non-minimal gravitational terms in L_m, are such as to make the theory multiplicatively renormalizable in curved spacetime. We will consider only spacetimes of constant curvature, excluding terms of the form $\phi\Box R$, etc., from the Lagrangian (1). Finally, $\lambda_i = \{M^2, g, h, \xi, \ldots, \alpha_{10}\}$ are all coupling constants whose dimensionality is clear from the form of the Lagrangian (1).

One-loop divergences of the model (1) are found in Ref. [6]. They yield the following running coupling constants (we give here their explicit expressions in the massless sector only):

$$
\begin{aligned}
g^2(t) &= g^2 B^{-1}(t), \qquad B(t) = 1 + \frac{3g^2 t}{2(4\pi)^3}, \\
\xi(t) &= \frac{1}{5} + \left(\xi - \frac{1}{5}\right) B^{-5/9}(t), \\
\eta_1(t) &= \eta_1 B^{1/18}(t), \\
\eta_3(t) &= B^{1/18}(t) \left[\eta_3 - \frac{1}{1200g}\left(B^{4/9}(t) - 1\right) + \frac{1}{5g}\left(\xi - \frac{1}{5}\right)\left(B^{-1/9}(t) - 1\right) \right. \\
&\quad \left. + \frac{1}{2g}\left(\xi - \frac{1}{5}\right)^2 \left(B^{-2/3}(t) - 1\right) \right], \\
\eta_{4,5}(t) &= B^{1/18}(t) \left[\eta_{4,5} \pm \frac{1}{120g}\left(B^{4/9}(t) - 1\right) \right], \\
h(t) &= h B^{1/18}(t).
\end{aligned} \tag{2}
$$

It is clear from expression (2) that the theory is asymptotically free at high energies ($g^2(t) \to 0$), and that it is asymptotically conformal invariant in the matter sector (see [8] for a review). From the complete set of one-loop divergences, given explicitly in Ref. [6], there are no problems in writing down all running coupling constants, including $M \neq 0$. To save space we have listed only those needed in this section.

Working with the massless version of the theory (1) we use (2) first to find the effective potential at one-loop and second to find RG improved asymptotic forms of this potential. We start by writing the effective action of this theory as:

$$\Gamma = \Gamma|_{\phi=0} + \int d^6x \sqrt{g}\, V + \cdots, \tag{3}$$

where the first term is the vacuum energy, the second is the effective potential. Terms that have not been explicitly included provide non-constant ϕ contributions to Γ. The multiplicative renormalizability of the theory guarantees that the effective action as well as the effective potential satisfies the RG equations:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_{\lambda_i} \frac{\partial}{\partial \lambda_i} - \gamma_\phi \phi \frac{\partial}{\partial \phi} \right) V = 0, \tag{4}$$

where γ_ϕ is the γ-function of the scalar field (computed here from [6]),

$$\gamma_\phi = \frac{g^2}{12(4\pi)^3}. \tag{5}$$

In order to find the effective potential as an expansion over curvature invariants, we will write the classical potential as (its form is clear from (1))

$$V^{(0)} = \sum_i V_i^{(0)}, \qquad V_i^{(0)} = a_i \lambda_i P_i \varphi^{k_i}, \tag{6}$$

where the a_i are numerical multipliers, $k_i \geq 1$ are integers, and the P_i are curvature invariants. Applying the method described in Ref. [9] (see also [7]), we can solve the RG equations (4) for a potential of the form (6). Restricting ourselves to one-loop and using the tree level potential (6) as boundary condition, we find (we skip technical details):

$$\begin{aligned} V \;=\; & \frac{1}{6} g\phi^3 - \frac{g^3\phi^3}{12(4\pi)^3}\left(\ln\frac{\phi^2}{\mu^2} - \frac{11}{3}\right) + h\phi + \frac{\xi}{2}R\phi^2 - \frac{g^2}{4(4\pi)^3}\left(\xi - \frac{1}{6}\right) R\phi^2 \left(\ln\frac{\phi^2}{\mu^2} - 3\right) \\ & + \eta_1 R\phi + \eta_3 R^2\phi + \eta_4 R_{\mu\nu}^2 \phi + \eta_5 R_{\mu\nu\alpha\beta}^2 \phi \\ & - \frac{g\phi}{(4\pi)^3}\left(\ln\frac{\phi^2}{\mu^2} - 2\right)\left[\frac{1}{4}\left(\xi - \frac{1}{6}\right)^2 R^2 - \frac{1}{360}R_{\mu\nu}^2 + \frac{1}{360}R_{\mu\nu\alpha\beta}^2\right]. \end{aligned} \tag{7}$$

This is the one-loop effective potential up to terms quadratic in the curvature. It is clear that this potential is not bounded from below (this is the well-known instability of the φ_6^3-theory). This kind of potential is useful for studying six-dimensional cosmology coupled to a φ_6^3 quantum field.

Another applications of the RG equations to the effective potential V, is to study the asymptotics of the effective potential in curved spacetime [10, 7]. The homogeneity condition of V has the form

$$V\left(e^{2t}\phi, e^{d_{\lambda_i} t}\lambda_i, e^{-2t}g_{\alpha\beta}, e^t\mu\right) = e^{6t} V\left(\phi, \lambda_i, g_{\alpha\beta}, \mu\right), \tag{8}$$

where $t = \text{const.}$ and d_{λ_i} is the dimension of λ_i. Relation (8) leads to the following equations:

$$\left(\partial_t + \mu\partial_\mu + d_{\lambda_i}\lambda_i\partial_{\lambda_i} + 2\phi\partial_\phi - 6\right) V\left(\phi, e^{-2t}g_{\alpha\beta}, \ldots\right) = 0, \tag{9}$$

$$\left(\partial_t + \mu\partial_\mu + d_{\lambda_i}\lambda_i\partial_{\lambda_i} - 2g_{\alpha\beta}\frac{\delta}{\delta g_{\alpha\beta}} - 6\right) V\left(e^{2t}\phi, g_{\alpha\beta}, \ldots\right) = 0, \tag{10}$$

where the parameters of the potential that are not written explicitly are not scaled. Combining Eq. (4) with Eqs. (9) and (10), we obtain the following:

$$\left[\partial_t - \left(\beta_{\lambda_i} - d_{\lambda_i}\lambda_i\right)\partial_{\lambda_i} + \left(\gamma_\phi + 2\right)\phi\partial_\phi - 6\right] V\left(\phi, e^{-2t}g_{\alpha\beta}, \ldots\right) = 0, \tag{11}$$

$$\left\{\partial_t + \left(1 + \gamma_\phi/2\right)^{-1}\left[-\left(\beta_{\lambda_i} - d_{\lambda_i}\lambda_i\right)\partial_{\lambda_i} - 2g_{\alpha\beta}\frac{\delta}{\delta g_{\alpha\beta}} - 6\right]\right\} V\left(e^{2t}\phi, g_{\alpha\beta}, \ldots\right) = 0. \tag{12}$$

The RG equations (11) and (12) describe the asymptotics of the effective potential. In particular, when $g_{\alpha\beta} \to e^{-2t}g_{\alpha\beta}$, $R^2 \to e^{4t}R^2$, Eq. (11) gives the asymptotic behavior of the effective potential in a strong gravitational field. Similarly, Eq. (12) gives the behavior of V in the case of a strong scalar field. Solving Eq. (11) we get (see also [10, 7]):

$$\begin{aligned} &V\left(\phi, e^{-2t} g_{\alpha\beta}, \lambda_i\right) = e^{6t} V\left(\phi(t), g_{\alpha\beta}, \lambda_i(t)\right), \\ &\dot{\lambda}_i(t) = \beta_{\lambda_i}(t) - d_{\lambda_i}\lambda_i(t), \qquad \lambda_i(0) = \lambda_i, \\ &\dot{\phi}(t) = -\left[2 + \gamma_\phi(t)\right]\phi(t), \qquad \phi(0) = \phi. \end{aligned} \tag{13}$$

Selecting the leading coupling constants from (2) and using (5) we obtain

$$V\left(\phi, e^{-2t} g_{\alpha\beta}, \lambda_i\right) \sim e^{6t}\phi(t)\left[\eta_3(t)R^2 + \eta_4(t)R^2_{\mu\nu} + \eta_5(t)R^2_{\mu\nu\alpha\beta}\right], \tag{14}$$

where

$$\phi(t) = \phi e^{-2t} B^{-1/18}(t). \tag{15}$$

Thus, the asymptotics of the effective potential in a strong gravitational field are defined by the non-minimal interaction of the scalar with the quadratic curvature invariants. Such approximations can be useful in studying quantum effects in the early universe (e.g. in the Kaluza-Klein framework).

In a similar way, we can solve Eq. (12), with the result

$$V\left(e^{2t}\phi, g_{\alpha\beta}, \lambda_i\right) = \exp\left[6\int_0^t dt'\, A(t')\right] V\left(\phi, g_{\alpha\beta}(t), \tilde{\lambda}_i(t)\right), \tag{16}$$

where

$$\begin{aligned} &A(t) = \left[1 + \frac{\gamma_\phi(t)}{2}\right]^{-1}, \qquad \phi(t) = \phi, \\ &\dot{g}_{\alpha\beta}(t,x) = 2A(t) g_{\alpha\beta}(t,x), \qquad g_{\alpha\beta}(0,x) = g_{\alpha\beta}(x), \\ &\dot{\tilde{\lambda}}_i(t) = A(t)\left[\beta_{\tilde{\lambda}_i}(t) - d_{\tilde{\lambda}_i}\tilde{\lambda}_i(t)\right]. \end{aligned} \tag{17}$$

As we see, contrary to what happens with Eqs. (13) for the effective couplings, the multiplier $A(t)$ appears on the rhs of Eqs. (17). Using arguments similar to the ones given in Ref. [11] (where the procedure to study the asymptotics of the effective potential in flat spacetime was developed), one can show that the presence of $A(t)$ does not influence the asymptotics of the effective couplings. Again, due to the fact that the theory is asymptotically free, it is natural to expect that the asymptotic behavior of the effective potential is given by the lowest order of perturbation theory, with the parameters replaced by the corresponding effective couplings.

Now, since $\phi(t) = \phi$, and the effective curvature is always small, $R(t) \sim e^{-2t}$ (see (17)), we get

$$V\left(e^{2t}\phi, g_{\alpha\beta}, \lambda_i\right) = \frac{1}{6} e^{6t} g(t)\phi^3. \tag{18}$$

The asymptotic value of the effective potential, in the limit of strong scalar curvature, is not bounded from below. This result can be useful for the study of six-dimensional quantum cosmology near the initial singularity. We conclude this discussion of the application of RG equations to the effective potential for the curved spacetime ϕ_6^3-theory and go on to an application the RG equations to the vacuum energy.

III. The vacuum energy in the ϕ_6^3-theory on a Kaluza-Klein spacetime.

Starting from the works [12] and [13], the vacuum energy of matter and gravitational fields on spherically compactified internal spaces was calculated and the process of quantum spontaneous compactification was studied. For a review and a list of references of papers on related question concerning Kaluza-Klein theories, see [14, 8]. In particular, in [15]-[20] and [21] vacuum energies were evaluated for scalar fields etc. (including gravity) defined on even-dimensional compactified spaces. In most of these studies only the divergent parts (in dim-reg) of the vacuum energies were evaluated.

Our goal here is to obtain the RG improved one-loop vacuum energies corresponding to the theory (1) on two Kaluza-Klein backgrounds, namely $R^4 \times S^1 \times S^1$ and $R^4 \times S^2$, and to investigate the process of spontaneous compactification.

(a) $R^4 \times S^1 \times S^1$ space.

At the one-loop level, the vacuum energy is given by

$$\Gamma^{(1)} = \frac{1}{2}\mathrm{Tr}\,\ln\left(-\Box + M^2\right). \tag{19}$$

The calculation can be done with the help of zeta function regularization (for an introduction, see [22]). The spectrum has the form

$$\lambda = k_4^2 + \left(\frac{2\pi n_1}{L_1}\right)^2 + \left(\frac{2\pi n_2}{L_2}\right)^2 + X, \tag{20}$$

with $X = M^2$ here, and the corresponding 'Euclideanized' zeta function is

$$\begin{aligned}
\zeta_E(s) &= \sum_\lambda \lambda^{-s} = \frac{1}{\Gamma(s)}\sum_\lambda \int_0^\infty dt\, t^{s-1} e^{-\lambda t} \\
&= \frac{1}{\Gamma(s)}\sum_{n_1 n_2}\int \frac{d^4k}{(2\pi)^4}\int_0^\infty dt\, t^{s-1}\exp\left\{-\left[k^2 + \left(\frac{2\pi n_1}{L_1}\right)^2 + \left(\frac{2\pi n_2}{L_2}\right)^2 + X\right]t\right\} \\
&= \frac{1}{4(2\pi)^{2(s+1)}(s-1)(s-2)}\left[\left(\frac{X}{4\pi^2}\right)^{2-s} + E\left(s-2; L_1^{-2}, 0, L_2^{-2}; \frac{X}{4\pi^2}\right)\right]. \qquad (21)
\end{aligned}$$

$E(s;a,b,c;q)$ is the zeta function introduced and studied in [23],

$$E(s;a,b,c;q) \equiv \sum_{m,n\in\mathbf{Z}}{}'(am^2 + bmn + cn^2 + q)^{-s}, \quad \mathrm{Re}\,(s) > 1. \tag{22}$$

In the general theory [23], one requires that $a, c > 0$, that the discriminant

$$\Delta = 4ac - b^2 = \left(\frac{2}{L_1 L_2}\right)^2 > 0, \tag{23}$$

and that $am^2 + bmn + cn^2 + q \neq 0$, for all $m, n \in \mathbf{Z}$. These conditions are all satisfied in this case. The analytic continuation [24] of this zeta function is:

$$\begin{aligned} E(s;a,b,c;q) = -q^{-s} + \frac{2\pi q^{1-s}}{(s-1)\sqrt{\Delta}} + \frac{4}{\Gamma(s)} \Bigg[\left(\frac{q}{a}\right)^{1/4} \left(\frac{\pi}{\sqrt{qa}}\right)^s \sum_{k=1}^{\infty} k^{s-1/2} K_{s-1/2}\left(2\pi k \sqrt{\frac{q}{a}}\right) \\ + \sqrt{\frac{q}{a}} \left(2\pi \sqrt{\frac{a}{q\Delta}}\right)^s \sum_{k=1}^{\infty} k^{s-1} K_{s-1}\left(4\pi k \sqrt{\frac{aq}{\Delta}}\right) \qquad (24) \\ + \sqrt{\frac{2}{a}} (2\pi)^s \sum_{k=1}^{\infty} k^{s-1/2} \cos(\pi k b/a) \sum_{d|k} d^{1-2s} \left(\Delta + \frac{4aq}{d^2}\right)^{1/4-s/2} K_{s-1/2}\left(\frac{\pi k}{a}\sqrt{\Delta + \frac{4aq}{d^2}}\right) \Bigg]. \end{aligned}$$

This explicit form (24) and its derivative (given below) appeared for the first time in [24]. It is remarkable that the only simple pole ($s = 1$) is so explicit in (24). This expression also has excellent convergence properties, in fact, for large q the convergence behavior of the series of Bessel functions is at least exponential. Particular values for $s = -n$, $n = 0, 1, 2, 3, \ldots$ are :

$$E(-n;a,b,c;q) = -q^n - \frac{2\pi}{n+1} \frac{q^{n+1}}{\sqrt{\Delta}}, \tag{25}$$

and

$$E(-n;a,b,c;0) = 0. \tag{26}$$

For the corresponding derivative at zero we have

$$\begin{aligned} E'(0;a,b,c;q) &= -\frac{2\pi q}{\sqrt{\Delta}} + \left(1 + \frac{2\pi q}{\sqrt{\Delta}}\right) \ln q \\ &\quad - 2\ln\left(1 - e^{-2\pi\sqrt{q/a}}\right) + 4\sqrt{\frac{q}{a}} \sum_{n=1}^{\infty} n^{-1} K_1\left(4n\pi \sqrt{\frac{aq}{\Delta}}\right) \qquad (27) \\ &\quad + 4 \sum_{n=1}^{\infty} n^{-1} \cos(n\pi b/a) \sum_{d|n} d \, \exp\left[-\frac{\pi n}{a}\left(\Delta + \frac{4aq}{d^2}\right)^{1/2}\right], \end{aligned}$$

and, in general, for $s = -n$, $n = 0, 1, 2, 3, \ldots$,

$$\begin{aligned} E'(-n;a,b,c;q) = -\frac{2\pi q^{n+1}}{(n+1)^2\sqrt{\Delta}} + q^n \left(1 + \frac{2\pi q}{(n+1)\sqrt{\Delta}}\right) \ln q \\ + 4\frac{(-1)^n}{n!\,\pi^n} \Bigg[q^{n/2+1/4} a^{n/2-1/4} \sum_{k=1}^{\infty} k^{-n-1/2} K_{n+1/2}\left(2\pi k \sqrt{\frac{q}{a}}\right) \\ + 2^{-n} \left(\frac{q}{a}\right)^{(n+1)/2} \Delta^{n/2} \sum_{k=1}^{\infty} k^{-n-1} K_{n+1}\left(4\pi k \sqrt{\frac{aq}{\Delta}}\right) \qquad (28) \\ + \frac{2^{-n+1/2}}{\sqrt{a}} \sum_{k=1}^{\infty} k^{-n-1/2} \cos(\pi k b/a) \sum_{d|k} d^{2n+1} \left(\Delta + \frac{4aq}{d^2}\right)^{k/2+1/4} K_{n+1/2}\left(\frac{\pi k}{a}\sqrt{\Delta + \frac{4aq}{d^2}}\right) \Bigg]. \end{aligned}$$

These are the only expressions needed for what follows. We want to evaluate the effective action $\Gamma^{(1)}/V_4$, where

$$\Gamma^{(1)} = \frac{1}{2}\left[{\zeta'}_E(0) + \zeta_E(0)\ln\mu^2\right], \tag{29}$$

and V_4 is the four-volume, $V_4 \equiv \int d^4x$. The result is immediate from the expressions above:

$$\begin{aligned}
\frac{\Gamma^{(1)}}{V_4} &= \frac{M^6 L_1 L_2}{128\pi^3}\left(-\frac{11}{36} + \frac{1}{6}\ln\frac{M^2}{\mu^2}\right) + 2\sqrt{\frac{2}{\pi}}\,\frac{M^{5/2}}{L_1^{3/2}}\sum_{n=1}^{\infty} n^{-5/2} K_{5/2}\left(nML_1\right) \\
&+ \frac{M^3 L_1}{4\pi L_2^2}\sum_{n=1}^{\infty} n^{-3} K_3\left(nML_1\right) \\
&+ \frac{2\sqrt{2}\,\pi^2}{L_1^{3/2}}\sum_{n=1}^{\infty} n^{-5/2}\sum_{d|n} d^5\left(\frac{4}{L_2^2} + \frac{M^2}{\pi^2 d^2}\right)^{5/4} K_{5/2}\left(\pi n L_1\sqrt{\frac{4}{L_2^2} + \frac{M^2}{\pi^2 d^2}}\right).
\end{aligned} \tag{30}$$

Notice that the result is given in terms of a rapidly convergent series, very well suited for numerical computation. In the massless case ($M^2 = 0$), we are left with the last term

$$\left.\frac{\Gamma^{(1)}}{V_4}\right|_{M^2=0} = \frac{16\,\pi^2}{L_1^{3/2} L_2^{5/2}}\sum_{n=1}^{\infty} n^{-5/2}\sigma_5(n)\, K_{5/2}\left(2\pi n\frac{L_1}{L_2}\right). \tag{31}$$

(b) $R^4 \times S^2$ space.

In this case, for simplicity, the vacuum energy will be calculated for the massless theory only

$$\Gamma^{(1)} = \frac{1}{2}\mathrm{Tr}\,\ln\left(-\Box + \xi R\right). \tag{32}$$

The spectrum is now

$$\lambda = k_4^2 - \Lambda_l^2 + X, \tag{33}$$

where $X = \xi R$. For the 2-sphere $R = 2/r^2$ when written in terms of the sphere's radius r. For scalar fields,

$$\Lambda_l^2 = -\frac{l(l+1)}{r^2}, \qquad l = 0, 1, 2, \dots \tag{34}$$

with associated multiplicities

$$D_l = 2l + 1. \tag{35}$$

The corresponding zeta function is

$$\begin{aligned}
\zeta_E(s) &= \frac{\Gamma(s-2)}{16\pi^2\Gamma(s)}\sum_l D_l(\Lambda_l^2 + X)^{2-s} \\
&= \frac{r^{2(s-2)}}{16\pi^2(s-1)(s-2)}\sum_{l=0}^{\infty}(2l+1)\left[(l+1/2)^2 + (Xr^2 - 1/4)\right]^{2-s} \\
&= -\frac{r^{2(s-2)}}{16\pi^2(s-1)(s-2)(s-3)}\,\frac{\partial}{\partial c}F(s-3; c; Xr^2 - 1/4)\bigg|_{c=1/2},
\end{aligned} \tag{36}$$

where $F(s;c;q)$ is another typical zeta function studied in full detail in [23],

$$F(s;c;q) \equiv \sum_{n=0}^{\infty} \left[(n+c)^2+q\right]^{-s} \equiv G(s;1,c;q). \tag{37}$$

From the general asymptotic expansion of $G(s;a,c;q)$ in powers of q^{-1} (see [23]),

$$G(s;a,c;q) \equiv \sum_{n=0}^{\infty} \left[a(n+c)^2+q\right]^{-s} \sim \frac{q^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+s)}{m!} \left(\frac{q}{a}\right)^{-m} \zeta_H(-2m,c) \tag{38}$$
$$+\sqrt{\frac{\pi}{a}} \frac{\Gamma(s-1/2)}{2\Gamma(s)} q^{1/2-s} + \frac{2\pi^s}{\Gamma(s)} a^{-1/4-s/2} q^{1/4-s/2} \sum_{n=1}^{\infty} n^{s-1/2} \cos(2\pi nc) K_{s-1/2}(2\pi n\sqrt{q/a}),$$

we easily obtain the asymptotic expansion:

$$\zeta_E(s) \sim -\frac{r^{2(s-2)}}{16\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(1-2^{1-2n}\right) B_{2n} \Gamma(s+n-3)}{n!\,\Gamma(s)} (Xr^2-1/4)^{3-s-n}, \tag{39}$$

where the B_{2n} are Bernoulli numbers. This yields immediately

$$\begin{aligned} \zeta_E(0) &= \frac{1}{16\pi^2 r^4}(Xr^2-1/4)^3 \left[-\frac{1}{6}+\frac{1}{24}(Xr^2-1/4)^{-1}-\frac{7}{480}(Xr^2-1/4)^{-2}\right. \\ &\qquad \left. +\frac{31}{8064}(Xr^2-1/4)^{-3}\right], \end{aligned} \tag{40}$$

and

$$\begin{aligned} \zeta'_E(0) &= -\zeta_E(0)\ln\left(X-\frac{1}{4r^2}\right)+\frac{1}{16\pi^2 r^4}(Xr^2-1/4)^3 \left[-\frac{11}{36}+\frac{1}{16}(Xr^2-1/4)^{-1}\right. \\ &\quad \left. -\frac{7}{480}(Xr^2-1/4)^{-2}+\sum_{n=4}^{\infty} \frac{(-1)^{n+1}\left(1-2^{1-2n}\right) B_{2n}}{n(n-1)(n-2)(n-3)} (Xr^2-1/4)^{-n}\right]. \end{aligned} \tag{41}$$

Finally,

$$\begin{aligned} \frac{\Gamma^{(1)}}{V_4} &= \frac{1}{32\pi^2 r^4}\left(2\xi-\frac{1}{4}\right)^3 \left\{\left[-\frac{11}{36}+\frac{1}{16}(2\xi-1/4)^{-1}\right.\right. \\ &\quad \left. -\frac{7}{480}(2\xi-1/4)^{-2}+\sum_{n=4}^{\infty} \frac{(-1)^{n+1}\left(1-2^{1-2n}\right) B_{2n}}{n(n-1)(n-2)(n-3)} (2\xi-1/4)^{-n}\right] \\ &\left. + \ln\left(\frac{\mu^2 r^2}{2\xi-1/4}\right)\left[-\frac{1}{6}+\frac{1}{24}(2\xi-1/4)^{-1}-\frac{7}{480}(2\xi-1/4)^{-2}+\frac{31}{8064}(2\xi-1/4)^{-3}\right]\right\}, \end{aligned} \tag{42}$$

which is an asymptotic expansion for large ξ, valid for $\xi > 1/8$. The optimal truncation of this asymptotic expansion is obtained after the $n = 4$ term. The point $\xi = 1/8$ has nothing to do with the conformal coupling value but instead depends on how the expansion of $\zeta_E(s)$ was done. This result ceases to be valid when $\xi \leq 1/8$. For the particular value $\xi = 1/8$, we can go back to the definition of the zeta function and show that F reduces to an ordinary Riemann zeta function: $F(s;1/2;0) = \zeta_H(2s;1/2) = (2^{2s}-1)\zeta(2s)$, and $\frac{\partial}{\partial c}\zeta_H(s;c) = -s\zeta_H(s+1;c)$, resulting in

$$\zeta_E(s) = \frac{r^{2(s-2)}(2^{2s-5}-1)}{8\pi^2(s-1)(s-2)}\zeta(2s-5), \qquad \xi = 1/8. \tag{43}$$

Then

$$\frac{\Gamma^{(1)}}{V_4} = \frac{1}{2^{11}\pi^3 r^4}\left\{\frac{31}{504}\left[\frac{3}{2} + \ln\left(\mu^2 r^2\right)\right] - \frac{\ln 2}{252} - 31\zeta'(-5)\right\}, \qquad \xi = 1/8. \tag{44}$$

For $\xi < 1/8$ the expansion (42) is replaced by a convergent series stemming from the binomial expansion of $F(s;,c;q)$ in powers of q, $0 \leq q < 1$ (see Eq. (37)). This can be easily done, as described in detail in [23, 24]. One would then have expressions which cover the whole range of values of ξ. However, in order to limit discussion we will illustrate the physical argument with the help of the $\xi > 1/8$ case, i.e., Eq. (42). Similar considerations would also apply to the $\xi \leq 1/8$ cases.

(c) Spontaneous compactification.

Using the vacuum energies that we have calculated above, we can now study the process of quantum spontaneous compactification on Kaluza-Klein backgrounds (see, e.g. [14]). In particular, we want to investigate consequences of using RG improvements to these energies on spontaneous compactification.

Turning back to our first example, we will now consider a ϕ_6^3 theory on a $R^4 \times S^1 \times S^1$ background where, for simplicity, we set $L_1 = L_2 = L$. The effective action which takes into account the one-loop corrections (30) is given by

$$\frac{\Gamma}{V_4} = -\Lambda L^2 + \frac{\Gamma^{(1)}}{V_4}. \tag{45}$$

The conditions of spontaneous compactification are:

$$\Gamma = 0, \qquad \frac{\partial}{\partial L}\left(\frac{\Gamma}{V_4}\right) = 0. \tag{46}$$

In the case under discussion, we have $\Gamma = \Gamma(\Lambda, M^2, \mu, L)$. Having two conditions and four parameters, the expectations of finding some solution of Eq. (46) are great. We will fix M^2 and μ and consider Γ as a function of the compactification length, L, only. In Fig. 1 we call this effective action (i.e., Γ divided by the four-volume) simply $V(L)$, and show its form explicitly for some specified values of M^2, μ and Λ that correspond to one of those situations in which spontaneous compactification takes place for a definite value of the compactification length L. Note that in this case we are beyond the range of validity of our approximation (which is analogous to that of the Coleman-Weinberg potential [2]), because of the large logarithmic contribution. For reasonably small values of the log (where the one-loop result can be trusted), that is up to $|\ln(M^2/\mu^2)| \simeq 1$, there is no minimum.

Let us now see how this picture changes, in general, when we take into account RG effects, e.g., when we enlarge the parameter space. As the theory is multiplicatively renormalizable, the effective action satisfies the RG equation (4). This equation can be solved using the method of the characteristics, yielding the so-called RG improved effective action (or Wilsonian effective action [25]). The corresponding RG improved effective potential [2] has been widely discussed in renormalizable theories with a Higgs sector, both in flat [3] and in curved spacetime [26].

The solution of the RG equation (4) (at $\phi = 0$) for the effective action gives

$$\Gamma(\lambda_i, g_{\alpha\beta}, \mu) = \Gamma(\lambda_i(t), g_{\alpha\beta}, \mu e^t), \tag{47}$$

where the effective couplings are given in (2). As boundary condition for (47) it is convenient to use the one-loop effective action (45). Then, the RG improved effective action is given by the same expression (45) but with the following changes of variables:

$$\begin{aligned} &M^2 \longrightarrow M^2(t) = M^2 \left(1 + \frac{3g^2 t}{2(4\pi)^3}\right)^{-5/9}, \\ &\mu^2 \longrightarrow \mu^2(t) = \mu^2 e^{2t}, \qquad \Lambda \longrightarrow \Lambda(t) = \Lambda - \frac{M^2}{6g^2}\left[\left(1 + \frac{3g^2 t}{2(4\pi)^3}\right)^{-2/3} - 1\right]. \end{aligned} \tag{48}$$

In order to define t we may choose the standard and most natural condition of dropping out the logarithmic term (for more details and different ways of defining t, see the last reference in [3])

$$M^2(t) = \mu^2 e^{2t}. \tag{49}$$

The solution of (49) determines the value of t as a function of g, M^2 and μ. Fixing M^2 and μ, as before, we obtain now the corresponding picture for the RG improved effective potential Γ as a function of L. This is depicted in Fig. 2 for specified values of M, μ, Λ and g that correspond to those of Fig. 1. The value of t which is a solution of Eq. (49) is here $t = 3.2182$. Differences in the effective potential Γ as a function of L caused by the RG improvement can be seen by comparing Figs. 1 and 2. A virtue of the RG improvement is that now it has physical sense to let the quocient M^2/μ^2 take such big values as in Fig. 1: the range of validity of the approximation is greatly enlarged. However, as is clearly observed in these figures, the RG improvement can modify the spontaneous compactification pattern dramatically. It actually happens in this case: the minimum disappears completely, at best it can turn into an inflection point (of the type of Fig. 3, although here it corresponds to the non-improved case). We see also that the renormalization group acting over an inflection point makes it evolve into the non-compactifying case (cf. Figs. 3 and 4). And this is all what happens in the model considered: the effect of the RG is destructive, concerning spontaneous compactification (it need not be so in other situations).

The same calculation can be repeated for the case of the RG improved effective action corresponding to the theory defined on the space $R^4 \times S^2$. Using the same principle as above, we can write

$$\frac{\Gamma}{V_4} = \frac{1}{V_4}\int d^6x \,\sqrt{g}\, L_{ext} + \frac{\Gamma^{(1)}}{V_4}, \tag{50}$$

where $L_{ext} = L_{ext}(\Lambda, k, \alpha_1, \ldots, \alpha_{10})$ (see Eq. (1)), and $\Gamma^{(1)}$ is given by (42). Expression (50) yields the one-loop effective action. The corresponding RG improved effective action is given by the same formula (50) but with the substitutions:

$$\xi \longrightarrow \xi(t) = \frac{1}{5} + \left(1 - \frac{1}{5}\right) B^{-5/9}(t), \qquad \alpha_i \longrightarrow \alpha_i(t), \tag{51}$$

where $i = 4, 5, \ldots, 10$. The explicit form of $\alpha_i(t)$ can be easily obtained from Refs. [6, 7]. One can establish the same comparison as before between the results of the spontaneous compactification process corresponding to the one-loop and to the RG improved effective actions on $R^4 \times S^2$. In this case there are a total of 14 parameters to satisfy just two conditions. Setting aside exceptional situations, an enormous variety of possibilities occur. For this case the difference in the process of compactification introduced by the RG improvement of the effective action is found again. We omit plots similar to ones corresponding to the previous example.

IV. Conclusions.

In this work we have investigated renormalization group effects in the ϕ_6^3 curved spacetime theory. Using this case as an example, the usefulness of the RG improvement in higher-dimensional theories has been demonstrated by calculating the one-loop effective potential and its asymptotics in strong fields. Additionally using the one-loop vacuum energy and the RG improved vacuum energy on Kaluza-Klein backgrounds $R^4 \times S^1 \times S^1$ and $R^4 \times S^2$, we have shown that the RG improvement can lead to significant qualitative differences in the spontaneous compactification of these theories. Such results clearly show that, in general, one-loop predictions in Kaluza-Klein theories should not always be trusted.

There are not many theories which are renormalizable in the standard way in higher dimensions. But the number increases by the introduction of higher-derivative kinetic terms (probably paying the price of spoiling unitary). For example, in $D = 6$ one can consider a gauge theory with $F_{\mu\nu}^3$-terms as kinetic terms plus any other term not prohibited by dimensional arguments (they can have equal or lower dimensionality) or gauge invariance. Such a theory will be renormalizable in the same sense as R^2-gravity in $D = 4$ is (see [8] for a review).

On the other hand, one can consider Kaluza-Klein theories to be renormalizable in the sense of the inclusion of an infinite number of additional terms and their corresponding counterterms (for a recent discussion, see [27]). Under these circumstances, the RG analysis can be applied again, but, of course, the RG equations will be infinite in number. Nevertheless, there are ways of truncating them in a systematic and consistent way, keeping just a finite number of terms. This can be done by considering, say just one-loop effects (in the even-dimensional case), or terms up to some particular order in the derivatives. As we have shown here, if the goal is studying spontaneous quantum compactification, the RG improved vacuum energy should be used.

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Figure captions

Fig. 1: The effective action $V \equiv \Gamma/V_4$, as a function of L, for the specified values of M^2, μ and Λ (in units of 10^4). They correspond to a situation in which spontaneous compactification takes place. At the minimum, the value of the compactification length L is selected.

Fig. 2: Plot of the renormalization group improved effective action for values of M, μ, Λ and g that correspond to those of Fig. 1. When comparing with Fig. 1, it is clearly observed that the RG improvement can modify the spontaneous compactification pattern.

Fig. 3: Plot of the effective action for values of M, μ, Λ and g that yield an inflection point, a situation that stays in the verge of spontaneous compactification.

Fig. 4: Plot of the RG improved effective action for values of M, μ, Λ and g corresponding to Fig. 3. The inflection point has disappeared and we are driven far away from spontaneous compactification.