Distance-Redshift in Inhomogeneous $\Omega_0 = 1$ Friedmann-Lemaître-Robertson-Walker Cosmology

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ABSTRACT

Distance–redshift relations are given in terms of associated Legendre functions for partially filled beam observations in spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies. These models are dynamically pressure-free, flat FLRW on large scales but, due to mass inhomogeneities, differ in their optical properties. The partially filled beam area-redshift equation is a Lame$'$ equation for arbitrary FLRW and is shown to simplify to the associated Legendre equation for the spatially flat, i.e., $\Omega_0 = 1$ case. We fit these new analytic Hubble curves to recent supernovae (SNe) data in an attempt to determine both the mass parameter Ω_m and the beam filling parameter ν. We find that current data are inadequate to limit ν. However, we are able to estimate what limits are possible when the number of observed SNe is increased by factor of 10 or 100, sample sizes achievable in the near future with the proposed SuperNova Acceleration Probe satellite.

Subject headings: cosmology: theory – large-scale structure of universe

1. INTRODUCTION

Distance-redshift or equivalently the Hubble curve is critical in determining current values of the cosmological parameters H_0, Ω_m, and Ω_Λ. Conversely, current values of these three parameters determine the large scale dynamics of the Universe into the distant past. A complication occurs when attempting to determine these parameters from high z comparisons to the standard Hubble curve. The standard Hubble curve is a theoretical quantity computed assuming all gravitating matter is homogeneously distributed; whereas, observational data is taken in the real inhomogeneous Universe. In an inhomogeneous universe an observing light beam is lensed by inhomogeneities located external to, but near the light beam, and defocused (relative to the standard Hubble curve) by the less than average matter density within the beam. The simplest way to take into account these effects is to correct all beams for the missing homogeneous matter but correct for lensing only when necessary. This procedure requires the introduction of one additional parameter, e.g., a filling parameter ν, $0 \le \nu \le 2$ defined by the fraction of inhomogeneous matter $\rho_I/\rho_0 \equiv \nu(\nu+1)/6 \le 1$ excluded from observing beams ($\nu = 0$ is the standard 100% filled beam FLRW case and $\nu = 2$ is the empty beam case). When observing high z objects ($z \sim 1$) the reader can think of the parameter ν as representing matter that exists in galaxies but not in the intergalactic medium. To find the theoretical Hubble curve for observations in such a universe one must solve the geometrical optics equation [see Kantowski (1998)] given as equation (1) in the next section. This equation is actually equivalent to the Lame$'$ equation for general FLRW but as pointed out by Kantowski et al. (2000) reduces to the associated Legendre equation (7) for the special case considered here, $\Omega_0 = 1$. In § 2 we solve this equation using appropriate boundary conditions and give the Hubble curve in terms of associated Legendre functions (eq.[10]) as well as in terms of hypergeometric functions (eq.[16]). In § 3 we fit this new Hubble curve to data for 60 supernovae (SNe) from the Supernova Cosmological Project (SCP) and from the Cala$'$n/Tololo Supernova Survey (CTSS) in an attempt to determine the mass parameter Ω_m and the filling parameter ν. In § 4 we give some concluding remarks.

2. The Luminosity Distance-redshift Relation

For models being discussed here (and for most cosmological models), angular or apparent size distance is related to luminosity distance by $D_<(z) = D_\ell(z)/(1+z)^2$. We choose to give luminosity distances in this paper. The $D_\ell(z)$ which accounts for a partially depleted mass density in the observing beam but neglects lensing by external masses is found by integrating the second order differential equation for the cross sectional area $A(z)$ of an observing beam from source ($z = z_s$) to observer ($z = 0$), see Kantowski (1998) for some history of this equation,[1]:

$$
\begin{aligned}
&(1+z)^3\sqrt{1+\Omega_m z+\Omega_\Lambda[(1+z)^{-2}-1]}\,\times \\
&\qquad \frac{d}{dz}(1+z)^3\sqrt{1+\Omega_m z+\Omega_\Lambda[(1+z)^{-2}-1]}\,\frac{d}{dz}\sqrt{A(z)} \\
&\qquad\qquad +\frac{(3+\nu)(2-\nu)}{4}\Omega_m(1+z)^5\sqrt{A(z)} = 0. \qquad (1)
\end{aligned}
$$

[1] This equation follows from applying Sach's optics equations [Sachs (1961)] to an inhomogeneous FLRW universe and neglecting shear (external lensing) Kantowski (1969). The first version of equation (1) was given by Zel'dovich (1964) and later Dyer & Roeder (1974) included the cosmological term.

The required boundary conditions are

$$
\begin{aligned}
\sqrt{A}|_s &= 0, \\
\left.\frac{d\sqrt{A}}{dz}\right|_s &= -\sqrt{\delta\Omega}\frac{c}{H_s(1+z_s)},
\end{aligned} \tag{2}
$$

where $\delta\Omega$ is the solid angle of the beam at the source and the FLRW value of the Hubble parameter at z_s is related to the current value H_0 at $z = 0$ by

$$
H_s = H_0(1+z_s)\ \sqrt{1+\Omega_m z_s + \Omega_\Lambda[(1+z_s)^{-2} - 1]}. \tag{3}
$$

The luminosity distance is then simply related to the area $A|_0$ of the beam at the observer by

$$
D_\ell^2 \equiv \frac{A|_0}{\delta\Omega}(1+z_s)^2. \tag{4}
$$

Equation (1) can be put into the form of a Heun equation and its solution has been given in terms of Heun functions in Kantowski (1998). Even though the Heun equation is only slightly more complicated than the hypergeometric equation, e.g., it has 4 regular singular points rather than 3, Heun functions are not yet available in standard libraries. Consequently, such expressions are not particularly useful for comparison with data, at this time. Because the exponents of three of the singular points of the area equation (in standard Heun form) are 0 and 1/2 [see eq. (13) in Kantowski (1998)], equation (1) is actually equivalent to the doubly periodic Lame$'$ equation. We now show that it reduces to the associated Legendre equation for the spatially flat universe, $\Omega_0 = 1$. The required change of dependent and independent variables are respectively

$$
P(A,z) \equiv (1+z)^{5/4}\sqrt{\frac{A}{\delta\Omega}}, \tag{5}
$$

$$
\eta(z) = \sqrt{\frac{1+\Omega_m z(3+3z+z^2)}{\Omega_m(1+z)^3}}. \tag{6}
$$

The resulting associated Legendre equation is

$$
(1-\eta^2)\frac{d^2P}{d\eta^2} - 2\eta\frac{dP}{d\eta} + \left(-\left[\frac{1}{6}\right]\left[\frac{5}{6}\right] - \frac{[(1+2\nu)/6]^2}{1-\eta^2}\right)\ P = 0\ , \tag{7}
$$

with initial conditions:

$$
\begin{aligned}
P|_s &= 0, \\
\left.\frac{dP}{d\eta}\right|_s &= \frac{c}{H_0}\frac{2}{3}\frac{\sqrt{\Omega_m}}{1-\Omega_m}\ (1+z_s)^{11/4}.
\end{aligned} \tag{8}
$$

The resulting luminosity distance is then given by

$$
D_\ell(z_s) = (1+z_s)P(\eta(0)). \tag{9}
$$

Expressed as associated Legendre functions equation (9) becomes

$$D_\ell(\Omega_m, \Omega_\Lambda = 1 - \Omega_m, \nu; z) = \frac{c}{H_0} \frac{2\,\Gamma\left(\frac{5-2\nu}{6}\right)\Gamma\left(\frac{7+2\nu}{6}\right)(1+z)^{3/4}}{(1+2\nu)\sqrt{\Omega_m}}$$
$$\times\left[\mathrm{P}\,{}^{(1+2\nu)/6}_{-1/6}\left(\sqrt{\frac{1+\Omega_m z(3+3z+z^2)}{\Omega_m(1+z)^3}}\right)\mathrm{P}\,{}^{-(1+2\nu)/6}_{-1/6}\left(\frac{1}{\sqrt{\Omega_m}}\right)\right.$$
$$\left.-\mathrm{P}\,{}^{(1+2\nu)/6}_{-1/6}\left(\frac{1}{\sqrt{\Omega_m}}\right)\mathrm{P}\,{}^{-(1+2\nu)/6}_{-1/6}\left(\sqrt{\frac{1+\Omega_m z(3+3z+z^2)}{\Omega_m(1+z)^3}}\right)\right]. \tag{10}$$

In this expression the associated Legendre functions take on their their analytically continued values (i.e., the arguments are on the real axis and > 1). When the filling parameter has values $\nu = 0, 1,$ or 2, equation (10) reduces respectively to equations (22), (39) and (54), of Kantowski et al. (2000). Because the associated Legendre equation is a special type of the hypergeometric equation it is always possible to write associated Legendre functions in terms of hypergeometric functions. And because hypergeometric functions are the more universally available, these results are the more useful for most parameter values. That the hypergeometric result existed has independently been seen by Kantowski et al. (2000) and Damianski et al. (2000). That the area equation reduces to the associated Legendre equation for another special case ($\Lambda = 0$) has been known for some time, see Kantowski et al. (1995) and Seitz & Schneider (1994). The appropriate change of variables is

$$h(A,z) \equiv (1+z)\sqrt{\frac{A}{\delta\Omega}} = (1+z)^{-1/4}P, \tag{11}$$
$$\zeta(z) = \frac{\Omega_m}{1-\Omega_m}(1+z)^3 + 1 = \frac{\eta^2}{\eta^2-1}. \tag{12}$$

The resulting hypergeometric equation is

$$(1-\zeta)\zeta\,\frac{d^2h}{d\zeta^2} + \left(\frac{1}{2} - \frac{7}{6}\zeta\right)\frac{dh}{d\zeta} + \frac{(\nu)(\nu+1)}{36}\,h = 0\ , \tag{13}$$

with initial conditions

$$h_s = 0,$$
$$\left.\frac{dh}{d\zeta}\right|_s = -\frac{c}{H_s}\,\frac{1-\Omega_m}{3\Omega_m}\,(1+z_s)^{-2}. \tag{14}$$

The resulting luminosity distance is then given by

$$D_\ell(z_s) = (1+z_s)h(\zeta(0)). \tag{15}$$

Expressed in terms of hypergeometric functions equation (15) becomes

$$D_\ell(\Omega_m, \Omega_\Lambda = 1 - \Omega_m, \nu; z) = \frac{c}{H_0}\frac{(1+z)\ 2}{(1+2\nu)\Omega_m^{1/3}}\left[1+\Omega_m z(3+3z+z^2)\right]^{\frac{\nu}{6}}$$

$$\times\left\{{}_2F_1\left(-\frac{\nu}{6},\frac{3-\nu}{6};\frac{5-2\nu}{6};\frac{1-\Omega_m}{[1+\Omega_m z(3+3z+z^2)]}\right){}_2F_1\left(\frac{1+\nu}{6},\frac{4+\nu}{6};\frac{7+2\nu}{6};1-\Omega_m\right)\right.$$
$$-\ [1+\Omega_m z(3+3z+z^2)]^{-\frac{1+2\nu}{6}}\,{}_2F_1\left(-\frac{\nu}{6},\frac{3-\nu}{6};\frac{5-2\nu}{6};1-\Omega_m\right)\times$$
$$\left.{}_2F_1\left(\frac{1+\nu}{6},\frac{4+\nu}{6};\frac{7+2\nu}{6};\frac{1-\Omega_m}{[1+\Omega_m z(3+3z+z^2)]}\right)\right\}. \tag{16}$$

3. Prospects for constraining ν from high redshift SNe Ia

Recent cosmic microwave background observations strongly imply a spatially flat universe (de Bernardis et al. (2000), Roos & Harun-or-Rashid (2000)), which naturally motivates application of the new distance formulae presented in §2 to a set of standard candles to estimate ν. We use the 60 SNe Ia from the combined Calan/Tololo + Supernova Cosmology Project (CT+SCP) as presented in Riess et al. (1998), Perlmutter et al. (1999). Combining these data with those from the High-z SN search (Schmidt et al. (1998)) would bring the total to about 100 SNe. However, we shall see that this somewhat complicated task would not increase the numbers of SNe enough to noticeably improve the estimate.

Rather than subject the data to an in-depth Bayesian re-analysis with the additional beam filling parameter ν included, we merely use a χ^2 goodness-of-fit estimation. We assume an intrinsic SN absolute magnitude of $M_B = -19.33$ and $H_0 = 65$ km s^{-1} Mpc^{-1} and follow the same procedure as Wang (2000a) to recover results consistent with Perlmutter et al. (1999) when $\nu = 0$, i.e., for observations in a homogeneous universe. Then we employ the formulae presented here to obtain confidence contours in the Ω_m–ν plane, constraining Ω_Λ so that $\Omega_\Lambda = 1.0 - \Omega_m$. By proceeding this way we are assuming that inhomogeneous matter, e.g., galaxies, are sufficiently removed from the lines of site of the 60 SNe and that lensing is negligible.

Figure 1 presents the 68, 90 and 99% confidence contours for the fit (solid, long-dashed and short-dashed lines respectively). The results at $\nu = 0$ are clearly consistent with the Perlmutter et al. (1999) findings. Unfortunately, no value of ν can be ruled out with this sample because of its size and depth in redshift space. The best fit overall is at ($\Omega_m = 0.31$, $\nu = 0.0$).

A simple-minded way to estimate the number of additional SNe required to rule out any value of ν at 99% confidence is to amplify the contribution of each SN to χ^2 by some factor. This has the effect of simulating more SNe which are exactly like the real ones, and hence the actual best fit will not move, but the contours will shrink. Figures 2 and 3 are the same contours as in Figure 1, but with sample size increased by factors of 10 and 100 respectively. We see that by simply enlarging the CT+SCP sample 10 times will allow a $\nu = 2$ value to be ruled out at 99% confidence. A factor of 100 will allow a much larger range of ν to be excluded. These results are quite promising because a sample size of SNe Ia in the thousands extending to even higher redshifts should be possible with a ground based SNe pencil beam survey (Wang (2000b)) or a satellite mission such as the SuperNova Acceleration Probe (SNAP - http://snap.lbl.gov).

The fact that the current data at $z < 1$ do not rule out any value of ν is not surprising. Assuming ($\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$), the increase in distance modulus incurred by increasing ν from 0 to 2 at $z = 0.5$ is only 0.04 magnitudes (see Kantowski (1998)). However, at redshifts of $z = 1.0$, 1.5 and 1.7 to be achieved in the future by SNAP, the increases are 0.14, 0.27 and 0.32 magnitudes respectively. Lensing complications will begin to occur at these higher redshifts and will have to be corrected for and/or selected against {see Wambsganss et al. (1997), Premadi et al. (1998), Holz & Wald (1998), Metcalf & Silk (1999), Tomita et al. (1999), and Wang (1999)}. Such data are likely to provide limited leverage in determining (Ω_m, Ω_Λ), unless ν is properly constrained. Even though the most prevalent opinion is that $\nu = 2$, i.e., there is no significant intergalactic medium, there may be a sea of massive neutrinos etc., out there that makes $0 < \nu < 2$.

As observations reach $z \sim 3$ lensing by galaxies becomes even more important and details of mass inhomogeneities will significantly distort the Hubble curve. Distances given here are still useful even without exact knowledge of those inhomoheneities. A lower bound on the primary image magnification at a given redshift (relative to the mean) is given by $\mu_{cutoff} = D_\ell^2(\Omega_m, \Omega_\Lambda, \nu = 0; z)/D_\ell^2(\Omega_m, \Omega_\Lambda, \nu; z)$. This bound represents sources which, by chance, are not lensed. This number can be compared to histograms given in Wambsganss et al. (1998), Premadi et al. (2001), Holz & Wald (1998), and Bergström et al. (2000).

4. Conclusions

We have given useful forms for the luminosity distance in the currently relevant inhomogeneous $\Omega_0 = 1$ FLRW cosmologies. These cosmologies are all dynamically FLRW in the large but differ in how gravitating matter affects optical observations. A beam filling parameter ν, $0 \leq \nu \leq 2$ allows the matter to vary from completely transparent and homogeneous to completely inhomogeneous and exterior to any observing beam. In order to determine the values of the cosmic parameters (Ω_m, Ω_Λ) from Hubble curves at high redshift, the value of ν must also be constrained. For fixed values of (Ω_m, Ω_Λ), increasing ν from 0 (totally homogeneous universe) to 2 (totally clumped) increases the distance moduli of points on the Hubble curve, *especially at higher redshifts* as pointed out in §3. When observational error is taken into account, the problem of using standard candles at high redshift while ignoring ν to obtain Ω_Λ will become particularly confounding.

The current sample of SNe Ia at $z < 1$ fail to constrain the value of the beam-filling parameter ν. Samples 10 to 100 times larger than the current sample, and in the same redshift range, will constrain ν. In order to unambiguously determined (Ω_m,Ω_Λ) from even higher redshift observations like those planned in the future, the distance-enhancing effect of ν must be accounted for in the luminosity distance formulae.

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REFERENCES

Bergström, L., Goliath, M., Goobar, A., & Mörtsell, E 2000, A&A

Damianski, M., de Ritis, R., Marino, A. A., & Piedipalumbo, E. 2000, (astro-ph/0004376)

de Bernardis, P. et al. 2000, (astro-ph/0004404)

Dyer, C. C. & Roeder, R. C. 1974, ApJ, 189, 167

Holz, D. E. & Wald, R. M. 1998, Phys. Rev. D, 58, 063501

Kantowski, R. 1969, ApJ, 155, 89

Kantowski, R., Vaughan, T., & Branch, D. 1995, ApJ, 447, 35

Kantowski, R. 1998, ApJ, 507, 483

Kantowski, R., Kao, J. K., & Thomas, R. C. 2000, ApJ, 545 in press (astro-ph/0002334)

Metcalf, R. B. & Silk, J. 1999, ApJ, 519, L1

Perlmutter, S. et al. 1999, ApJ, 517, 565

Premadi, P., Martel, H. & Matzner, R. 1998, ApJ, 493, 10

Premadi, P., Martel, H., Matzner, R. & Futamase, T. 2001, ApJS, in press astro-ph/0101359

Riess, A. G. et al. 1998, AJ, 116, 1009

Roos, M. & Harun-or-Rashid, S. M. 2000, (astro-ph/0005541)

Sachs, R. K. 1961, Proc. R. Soc. London A, 264, 309

Schmidt, B. P. et al. 1998, ApJ, 507, 46

Seitz, S. & Schneider, P. 1994, A&A 287, 349

Tomita, K., Premadi, P., & Nakamura, T. T. 1999, Prog. Theor. Phys. Supp.

Wambsganss, J., Cen, R., Xu, G., & Ostriker, J. P. 1997, ApJ, 475, L81

Wambsganss, J., Cen, R., & Ostriker, J. P. 1998, ApJ, 494, 29

Wang, Y. 1999, ApJ, 525, 651

Wang, Y. 2000, ApJ, 536, 531

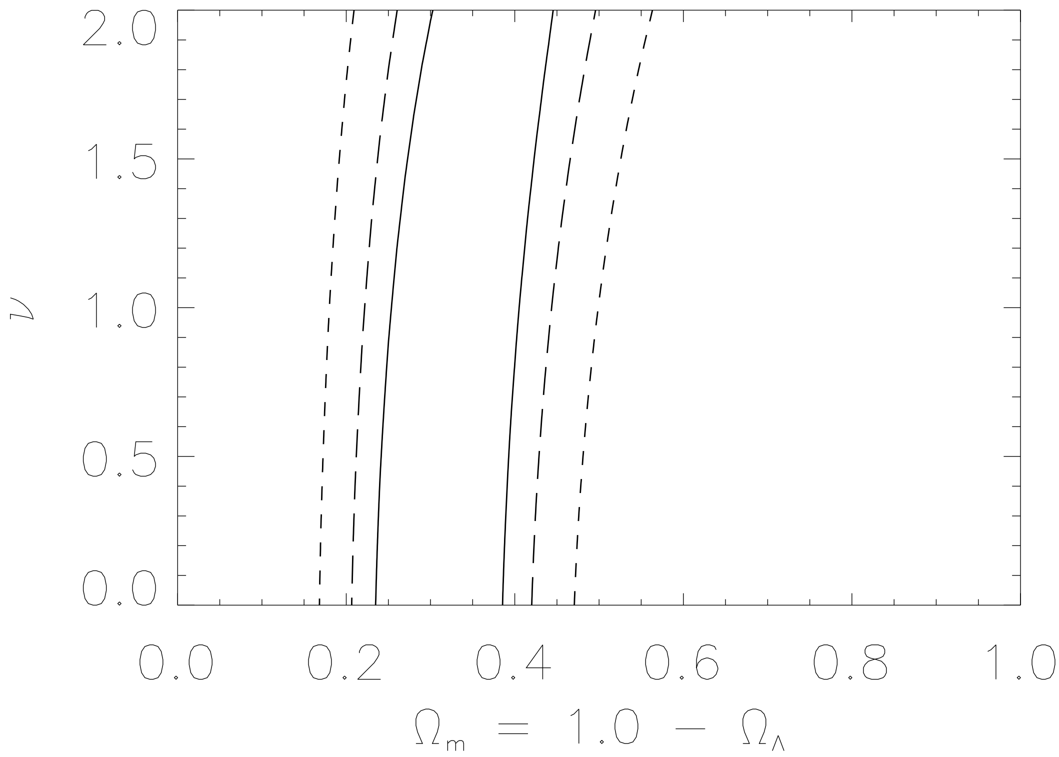
Wang, Y. 2000, ApJ, 531, 676

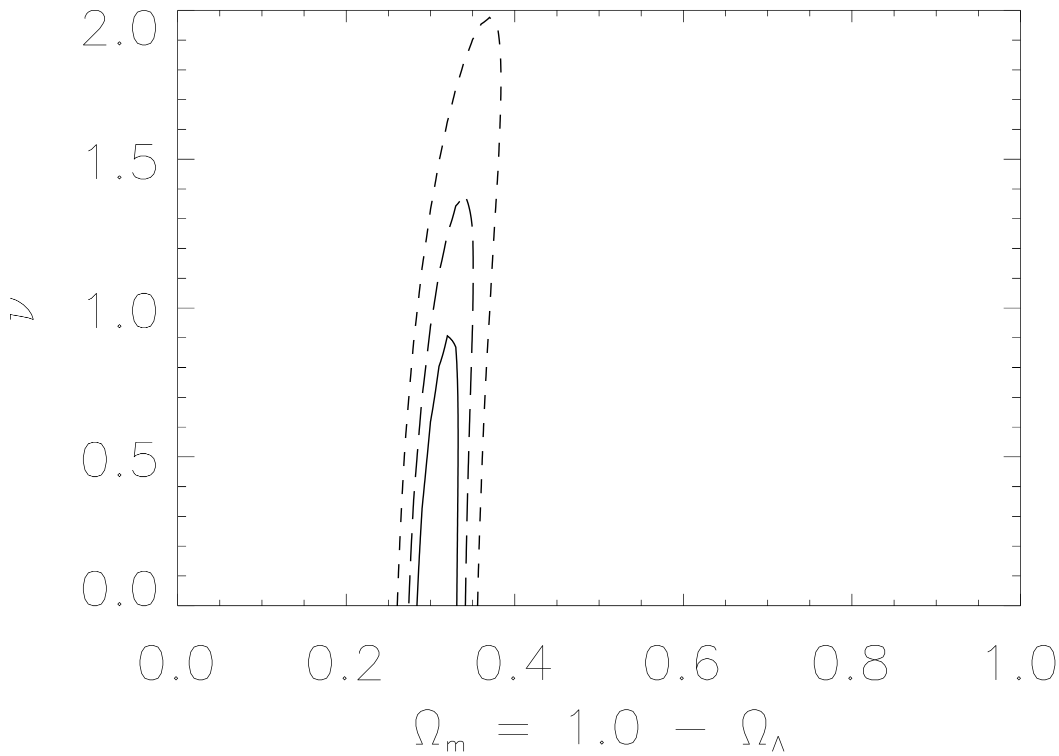
Zel'dovich, Ya. B. 1964, Soviet Ast.–AJ, 8, 13

Fig. 1.— The Ω_m–ν plane with 68, 90 and 99% confidence contours (solid, long-dashed and short-dashed, respectively) resulting from an attempt to constrain those parameters using the 60 SNe Ia from the Calan/Tololo Supernova Search + Supernova Cosmology Project sample Perlmutter et al. (1999). The data were first fit assuming $\nu = 0$ (completely homogeneous universe) to recover a result consistent with the original findings. Then assuming $\Omega_m = 1 - \Omega_\Lambda$ the above χ^2 grid was calculated and contours of equal $\Delta\chi^2$ above the minimum χ^2 were plotted.

Fig. 2.— Same as Fig. 1 except 10 times more SN measurements exactly like the 60 SNe Ia from the Calan/Tololo + Supernova Cosmology Project sample were used in computing the confidence levels.

Fig. 3.— Same as Fig. 2 except 100 rather than 10 times more SN were assumed to exist.




2.0
1.5
1.0
0.5
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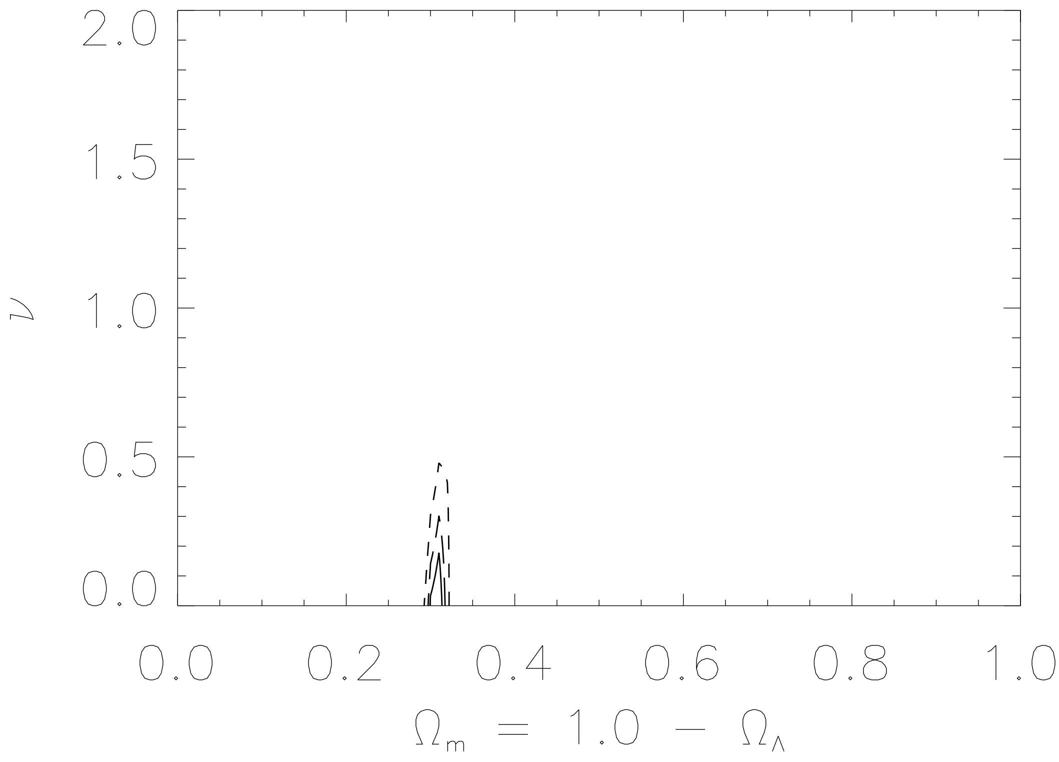
ν
0.0
0.2
0.4
0.6
0.8
1.0
Ωm = 1.0 − ΩΛ


2.0
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$\Omega_m = 1.0 - \Omega_\Lambda$