CHAPTER XII

THE GAMMA FUNCTION

12·1. *Definitions of the Gamma-function. The Weierstrassian product.*

Historically, the Gamma-function* $\Gamma(z)$ was first defined by Euler as the limit of a product (§ 12·11) from which can be derived the infinite integral $\int_0^\infty t^{z-1}e^{-t}dt$; but in developing the theory of the function, it is more convenient to define it by means of an infinite product of Weierstrass' canonical form.

Consider the product $\quad ze^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\}$,

where $$\gamma = \lim_{m\to\infty}\left\{\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{m}-\log m\right\}=0·5772157\dots.$$

[The constant γ is known as Euler's or Mascheroni's constant; to prove that it exists we observe that, if

$$u_n=\int_0^1 \frac{t}{n(n+t)}dt=\frac{1}{n}-\log\frac{n+1}{n},$$

u_n is positive and less than $\int_0^1 \frac{dt}{n^2}=\frac{1}{n^2}$; therefore $\sum_{n=1}^{\infty} u_n$ converges, and

$$\lim_{m\to\infty}\left\{\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{m}-\log m\right\}=\lim_{m\to\infty}\left\{\sum_{n=1}^{m}u_n+\log\frac{m+1}{m}\right\}=\sum_{n=1}^{\infty}u_n.$$

The value of γ has been calculated by J. C. Adams to 260 places of decimals.]

The product under consideration represents an analytic function of z, for all values of z; for, if N be an integer such that $|z|\leqslant\frac{1}{2}N$, we have†, if $n>N$,

$$\begin{aligned}\left|\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right| &= \left|-\frac{1}{2}\frac{z^2}{n^2}+\frac{1}{3}\frac{z^3}{n^3}-\dots\right| \\ &\leqslant \frac{|z|^2}{n^2}\left\{1+\left|\frac{z}{n}\right|+\left|\frac{z^2}{n^2}\right|+\dots\right\} \\ &\leqslant \frac{1}{4}\frac{N^2}{n^2}\left\{1+\frac{1}{2}+\frac{1}{2^2}+\dots\right\}\leqslant\frac{1}{2}\frac{N^2}{n^2}.\end{aligned}$$

Since the series $\sum_{n=N+1}^{\infty}\{N^2/(2n^2)\}$ converges, it follows that, when $|z|\leqslant\frac{1}{2}N$,

* The notation $\Gamma(z)$ was introduced by Legendre in 1814.

† Taking the principal value of $\log(1+z/n)$.

$\sum_{n=N+1}^{\infty} \left\{\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right\}$ is an absolutely and uniformly convergent series of analytic functions, and so it is an analytic function (§ 5·3); consequently its exponential $\prod_{n=N+1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\}$ is an analytic function, and so $ze^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\}$ is an analytic function when $|z| \leqslant \frac{1}{2}N$, where N is any integer; that is to say, the product is analytic for all finite values of z.

The Gamma-function was defined by Weierstrass* by the equation

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\};$$

from this equation *it is apparent that* $\Gamma(z)$ *is analytic except at the points* $z = 0, -1, -2, \ldots,$ *where it has simple poles.*

Proofs have been published by Hölder†, Moore‡, and Barnes§ of a theorem known to Weierstrass that the Gamma-function does not satisfy any differential equation with rational coefficients.

Example 1. Prove that

$$\Gamma(1) = 1, \quad \Gamma'(1) = -\gamma,$$

where γ is Euler's constant.

[Justify differentiating logarithmically the equation

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\}$$

by § 4·7, and put $z=1$ after the differentiations have been performed.]

Example 2. Shew that

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} = \int_0^1 \frac{1-(1-t)^n}{t}\,dt,$$

and hence that Euler's constant γ is given by||

$$\lim_{n\to\infty}\left[\int_0^1 \left\{1-\left(1-\frac{t}{n}\right)^n\right\}\frac{dt}{t} - \int_1^n \left(1-\frac{t}{n}\right)^n \frac{dt}{t}\right].$$

Example 3. Shew that

$$\prod_{n=1}^{\infty}\left\{\left(1-\frac{x}{z+n}\right)e^{\frac{x}{n}}\right\} = \frac{e^{\gamma x}\,\Gamma(z+1)}{\Gamma(z-x+1)}.$$

* *Journal für Math.* LI. (1856). This formula for $\Gamma(z)$ had been obtained from Euler's formula (§ 12·11) in 1848 by F. W. Newman, *Cambridge and Dublin Math. Journal*, III. (1848), p. 60.

† *Math. Ann.* XXVIII. (1887), pp. 1–13.

‡ *Math. Ann.* XLVIII. (1897), pp. 70–74.

§ *Messenger of Math.* XXIX. (1900), pp. 122–128.

|| The reader will see later (§ 12·2 example 4) that this limit may be written

$$\int_0^1 (1-e^{-t})\frac{dt}{t} - \int_1^{\infty}\frac{e^{-t}\,dt}{t}.$$

12·11. *Euler's formula for the Gamma-function.*

By the definition of an infinite product we have

$$\frac{1}{\Gamma(z)} = z\left[\lim_{m\to\infty} e^{\left(1+\frac{1}{2}+\dots+\frac{1}{m}-\log m\right)z}\right]\left[\lim_{m\to\infty}\prod_{n=1}^{m}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\}\right]$$

$$= z\lim_{m\to\infty}\left[e^{\left(1+\frac{1}{2}+\dots+\frac{1}{m}-\log m\right)z}\prod_{n=1}^{m}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\}\right]$$

$$= z\lim_{m\to\infty}\left[m^{-z}\prod_{n=1}^{m}\left(1+\frac{z}{n}\right)\right]$$

$$= z\lim_{m\to\infty}\left[\prod_{n=1}^{m-1}\left(1+\frac{1}{n}\right)^{-z}\prod_{n=1}^{m}\left(1+\frac{z}{n}\right)\right]$$

$$= z\lim_{m\to\infty}\left[\prod_{n=1}^{m}\left\{\left(1+\frac{z}{n}\right)\left(1+\frac{1}{n}\right)^{-z}\right\}\left(1+\frac{1}{m}\right)^{z}\right].$$

Hence
$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right\}.$$

This formula is due to Euler*; it is valid except when $z = 0, -1, -2, \dots$.

Example. Prove that

$$\Gamma(z) = \lim_{n\to\infty}\frac{1.2\dots(n-1)}{z(z+1)\dots(z+n-1)}n^{z}. \qquad \text{(Euler.)}$$

12·12. *The difference equation satisfied by the Gamma-function.*

We shall now shew that the function $\Gamma(z)$ satisfies the difference equation

$$\Gamma(z+1) = z\Gamma(z).$$

For, by Euler's formula, if z is not a negative integer,

$$\Gamma(z+1)/\Gamma(z) = \frac{1}{z+1}\left[\lim_{m\to\infty}\prod_{n=1}^{m}\frac{\left(1+\frac{1}{n}\right)^{z+1}}{1+\frac{z+1}{n}}\right] \div \left[\frac{1}{z}\lim_{m\to\infty}\prod_{n=1}^{m}\frac{\left(1+\frac{1}{n}\right)^{z}}{1+\frac{z}{n}}\right]$$

$$= \frac{z}{z+1}\lim_{m\to\infty}\prod_{n=1}^{m}\left\{\frac{\left(1+\frac{1}{n}\right)(z+n)}{z+n+1}\right\}$$

$$= z\lim_{m\to\infty}\frac{m+1}{z+m+1} = z.$$

This is one of the most important properties of the Gamma-function.

Since $\Gamma(1) = 1$, it follows that, if z is a positive integer, $\Gamma(z) = (z-1)!$.

* It was given in 1729 in a letter to Goldbach, printed in Fuss' *Corresp. Math.*

Example. Prove that

$$\frac{1}{\Gamma(z+1)}+\frac{1}{\Gamma(z+2)}+\frac{1}{\Gamma(z+3)}+\ldots=\frac{e}{\Gamma(z)}\left\{\frac{1}{z}-\frac{1}{1!}\frac{1}{z+1}+\frac{1}{2!}\frac{1}{z+2}-\ldots\right\}.$$

[Consider the expression

$$\frac{1}{z}+\frac{1}{z(z+1)}+\frac{1}{z(z+1)(z+2)}+\ldots+\frac{1}{z(z+1)\ldots(z+m)}.$$

It can be expressed in partial fractions in the form $\sum_{n=0}^{m}\frac{a_n}{z+n}$, where

$$a_n=\frac{(-)^n}{n!}\left\{1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{(m-n)!}\right\}=\frac{(-)^n}{n!}\left\{e-\sum_{r=m-n+1}^{\infty}\frac{1}{r!}\right\}.$$

Noting that $\sum_{r=m-n+1}^{\infty}\frac{1}{r!}<\frac{e}{(m-n+1)!}$, prove that $\sum_{n=0}^{m}\frac{(-)^n}{n!}\frac{1}{z+n}\left\{\sum_{r=m-n+1}^{\infty}\frac{1}{r!}\right\}\to 0$ as $m\to\infty$ when z is not a negative integer.]

12·13. *The evaluation of a general class of infinite products.*

By means of the Gamma-function, it is possible to evaluate the general class of infinite products of the form

$$\prod_{n=1}^{\infty} u_n,$$

where u_n is any rational function of the index n.

For, resolving u_n into its factors, we can write the product in the form

$$\prod_{n=1}^{\infty}\left\{\frac{A(n-a_1)(n-a_2)\ldots(n-a_k)}{(n-b_1)\ldots(n-b_l)}\right\};$$

and it is supposed that no factor in the denominator vanishes.

In order that this product may converge, the number of factors in the numerator must clearly be the same as the number of factors in the denominator, and also $A=1$; for, otherwise, the general factor of the product would not tend to the value unity as n tends to infinity.

We have therefore $k=l$, and, denoting the product by P, we may write

$$P=\prod_{n=1}^{\infty}\left\{\frac{(n-a_1)\ldots(n-a_k)}{(n-b_1)\ldots(n-b_k)}\right\}.$$

The general term in this product can be written

$$\left(1-\frac{a_1}{n}\right)\ldots\left(1-\frac{a_k}{n}\right)\left(1-\frac{b_1}{n}\right)^{-1}\ldots\left(1-\frac{b_k}{n}\right)^{-1}$$

$$=1-\frac{a_1+a_2+\ldots+a_k-b_1-\ldots-b_k}{n}+A_n,$$

where A_n is $O(n^{-2})$ when n is large.

In order that the infinite product may be absolutely convergent, it is therefore necessary further (§ 2·7) that

$$a_1+\ldots+a_k-b_1-\ldots-b_k=0.$$

We can therefore introduce the factor

$$\exp\{n^{-1}(a_1 + \ldots + a_k - b_1 - \ldots - b_k)\}$$

into the general factor of the product, without altering its value; and thus we have

$$P = \prod_{n=1}^{\infty} \left\{ \frac{\left(1 - \frac{a_1}{n}\right) e^{\frac{a_1}{n}} \left(1 - \frac{a_2}{n}\right) e^{\frac{a_2}{n}} \ldots \left(1 - \frac{a_k}{n}\right) e^{\frac{a_k}{n}}}{\left(1 - \frac{b_1}{n}\right) e^{\frac{b_1}{n}} \ldots \left(1 - \frac{b_k}{n}\right) e^{\frac{b_k}{n}}} \right\}$$

But it is obvious from the Weierstrassian definition of the Gamma-function that

$$\prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} \right\} = \frac{1}{-z\Gamma(-z)\, e^{-\gamma z}},$$

and so $$P = \frac{b_1 \Gamma(-b_1)\, b_2 \Gamma(-b_2) \ldots b_k \Gamma(-b_k)}{a_1 \Gamma(-a_1) \ldots a_k \Gamma(-a_k)} = \prod_{m=1}^{k} \frac{\Gamma(1 - b_m)}{\Gamma(1 - a_m)};$$

a formula which expresses the general infinite product P in terms of the Gamma-function.

Example 1. Prove that

$$\prod_{s=1}^{\infty} \frac{s(a+b+s)}{(a+s)(b+s)} = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+1)}.$$

Example 2. Shew that, if $a = \cos(2\pi/n) + i \sin(2\pi/n)$, then

$$x\left(1 - \frac{x}{1^n}\right)\left(1 - \frac{x}{2^n}\right) \ldots = \{-\Gamma(-x^{\frac{1}{n}})\,\Gamma(-ax^{\frac{1}{n}}) \ldots \Gamma(-a^{n-1}x^{\frac{1}{n}})\}^{-1}.$$

12·14. *Connexion between the Gamma-function and the circular functions.*

We now proceed to establish another most important property of the Gamma-function, expressed by the equation

$$\Gamma(z)\,\Gamma(1 - z) = \frac{\pi}{\sin \pi z}.$$

We have, by the definition of Weierstrass (§ 12·1),

$$\Gamma(z)\,\Gamma(-z) = -\frac{1}{z^2} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right\}^{-1} \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} \right\}^{-1}$$

$$= \frac{-\pi}{z \sin \pi z},$$

by § 7·5 example 1. Since, by § 12·12,

$$\Gamma(1 - z) = -z\Gamma(-z)$$

we have the result stated.

Corollary 1. If we assign to z the value $\frac{1}{2}$, this formula gives $\{\Gamma(\frac{1}{2})\}^2 = \pi$; since, by the formula of Weierstrass, $\Gamma(\frac{1}{2})$ is positive, we have

$$\Gamma(\tfrac{1}{2}) = \pi^{\frac{1}{2}}.$$

Corollary 2. If $\psi(z) = \Gamma'(z)/\Gamma(z)$, then $\psi(1-z) - \psi(z) = \pi \cot \pi z$.

12·15. *The multiplication-theorem of Gauss* and Legendre.*

We shall next obtain the result

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\dots\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{\frac{1}{2}(n-1)}\,n^{\frac{1}{2}-nz}\,\Gamma(nz).$$

For let
$$\phi(z) = \frac{n^{nz}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\dots\Gamma\left(z+\frac{n-1}{n}\right)}{n\,\Gamma(nz)}.$$

Then we have, by Euler's formula (§ 12·11 example),

$$\phi(z) = \frac{n^{nz}\prod\limits_{r=0}^{n-1}\lim\limits_{m\to\infty}\dfrac{1\,.\,2\dots(m-1)\,.\,m^{z+r/n}}{\left(z+\frac{r}{n}\right)\left(z+\frac{r}{n}+1\right)\dots\left(z+\frac{r}{n}+m-1\right)}}{n\lim\limits_{m\to\infty}\dfrac{1\,.\,2\dots(nm-1)\,.\,(nm)^{nz}}{nz\,(nz+1)\dots(nz+nm-1)}}$$

$$= n^{nz-1}\lim_{m\to\infty}\frac{\{(m-1)!\}^n\,m^{nz+\frac{1}{2}(n-1)}\,n^{mn}}{(nm-1)!\,(nm)^{nz}}$$

$$= \lim_{m\to\infty}\frac{\{(m-1)!\}^n\,m^{\frac{1}{2}(n-1)}\,n^{mn-1}}{(nm-1)!}.$$

It is evident from this last equation that $\phi(z)$ is independent of z. Thus $\phi(z)$ is equal to the value which it has when $z = \frac{1}{n}$; and so

$$\phi(z) = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right).$$

Therefore
$$\{\phi(z)\}^2 = \prod_{r=1}^{n-1}\left\{\Gamma\left(\frac{r}{n}\right)\Gamma\left(1-\frac{r}{n}\right)\right\}$$

$$= \frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots\sin\frac{(n-1)\pi}{n}} = \frac{(2\pi)^{n-1}}{n}.$$

Thus, since $\phi(n^{-1})$ is positive,

$$\phi(z) = (2\pi)^{\frac{1}{2}(n-1)}\,n^{-\frac{1}{2}},$$

i.e.
$$\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\dots\Gamma\left(z+\frac{n-1}{n}\right) = n^{\frac{1}{2}-nz}\,(2\pi)^{\frac{1}{2}(n-1)}\,\Gamma(nz).$$

Corollary. Taking $n=2$, we have

$$2^{2z-1}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) = \pi^{\frac{1}{2}}\,\Gamma(2z).$$

This is called the *duplication formula.*

* *Werke*, III. p. 149. The case in which $n=2$ was given by Legendre.

Example. If
$$B(p, q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$
shew that
$$B(np, nq)=n^{-nq}\frac{B(p, q)\,B\left(p+\frac{1}{n}, q\right)\dots B\left(p+\frac{n-1}{n}, q\right)}{B(q, q)\,B(2q, q)\dots B\{(n-1)q, q\}}.$$

12·16. *Expansions for the logarithmic derivates of the Gamma-function.*

We have
$$\{\Gamma(z+1)\}^{-1}=e^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\}.$$

Differentiating logarithmically (§ 4·7), this gives
$$\frac{d\log\Gamma(z+1)}{dz}=-\gamma+\frac{z}{1(z+1)}+\frac{z}{2(z+2)}+\frac{z}{3(z+3)}+\dots.$$

Therefore, since $\log\Gamma(z+1)=\log z+\log\Gamma(z)$, we have
$$\frac{d}{dz}\log\Gamma(z)=-\gamma-\frac{1}{z}+z\sum_{n=1}^{\infty}\frac{1}{n(z+n)}.$$

Differentiating again, $\frac{d^2}{dz^2}\log\Gamma(z+1)=\frac{d}{dz}\left\{\frac{z}{1(z+1)}+\frac{z}{2(z+2)}+\dots\right\}$
$$=\frac{1}{(z+1)^2}+\frac{1}{(z+2)^2}+\dots.$$

These expansions are occasionally used in applications of the theory.

12·2. *Euler's expression of $\Gamma(z)$ as an infinite integral.*

The infinite integral $\int_0^\infty e^{-t}t^{z-1}dt$ represents an analytic function of z when* the real part of z is positive (§ 5·32); it is called the *Eulerian Integral of the Second Kind*†. It will now be shewn that, when $R(z)>0$, the integral is equal to $\Gamma(z)$. Denoting the real part of z by x, we have $x>0$. Now, if ‡
$$\Pi(z, n)=\int_0^n\left(1-\frac{t}{n}\right)^n t^{z-1}dt,$$
we have
$$\Pi(z, n)=n^z\int_0^1(1-\tau)^n\tau^{z-1}d\tau,$$
if we write $t=n\tau$; it is easily shewn by repeated integrations by parts that, when $x>0$ and n is a positive integer,
$$\int_0^1(1-\tau)^n\tau^{z-1}d\tau=\left[\frac{1}{z}\tau^z(1-\tau)^n\right]_0^1+\frac{n}{z}\int_0^1(1-\tau)^{n-1}\tau^z d\tau$$
$$=\dots\dots\dots\dots$$
$$=\frac{n(n-1)\dots 1}{z(z+1)\dots(z+n-1)}\int_0^1\tau^{z+n-1}d\tau,$$
and so
$$\Pi(z, n)=\frac{1.2\dots n}{z(z+1)\dots(z+n)}n^z.$$

Hence, by the example of § 12·11, $\Pi(z, n)\to\Gamma(z)$ as $n\to\infty$.

* If the real part of z is not positive the integral does not converge on account of the singularity of the integrand at $t=0$.

† The name was given by Legendre; see § 12·4 for the Eulerian Integral of the First Kind.

‡ The many-valued function t^{z-1} is made precise by the equation $t^{z-1}=e^{(z-1)\log t}$, $\log t$ being purely real.

Consequently $$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt.$$

And so, if $$\Gamma_1(z) = \int_0^\infty e^{-t} t^{z-1} dt,$$
we have
$$\Gamma_1(z) - \Gamma(z) = \lim_{n\to\infty} \left[\int_0^n \left\{ e^{-t} - \left(1 - \frac{t}{n}\right)^n \right\} t^{z-1} dt + \int_n^\infty e^{-t} t^{z-1} dt \right].$$

Now $$\lim_{n\to\infty} \int_n^\infty e^{-t} t^{z-1} dt = 0,$$
since $\int_0^\infty e^{-t} t^{z-1} dt$ converges.

To shew that zero is the limit of the first of the two integrals in the formula for $\Gamma_1(z) - \Gamma(z)$ we observe that
$$0 \leqslant e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqslant n^{-1} t^2 e^{-t}.$$

[To establish these inequalities, we proceed as follows: when $0 \leqslant y < 1$,
$$1 + y \leqslant e^y \leqslant (1 - y)^{-1},$$
from the series for e^y and $(1-y)^{-1}$. Writing t/n for y, we have
$$\left(1 + \frac{t}{n}\right)^{-n} \geqslant e^{-t} \geqslant \left(1 - \frac{t}{n}\right)^n,$$
and so
$$\begin{aligned} 0 &\leqslant e^{-t} - \left(1 - \frac{t}{n}\right)^n \\ &= e^{-t}\left\{1 - e^t\left(1 - \frac{t}{n}\right)^n\right\} \\ &\leqslant e^{-t}\left\{1 - \left(1 - \frac{t^2}{n^2}\right)^n\right\}. \end{aligned}$$

Now, if $0 \leqslant a \leqslant 1$, $(1-a)^n \geqslant 1 - na$ by induction when $na < 1$ and obviously when $na \geqslant 1$; and, writing t^2/n^2 for a, we get
$$1 - \left(1 - \frac{t^2}{n^2}\right)^n \leqslant \frac{t^2}{n}$$
and so*
$$0 \leqslant e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqslant e^{-t} t^2/n,$$
which is the required result.]

From the inequalities, it follows at once that
$$\left| \int_0^n \left\{ e^{-t} - \left(1 - \frac{t}{n}\right)^n \right\} t^{z-1} dt \right| \leqslant \int_0^n n^{-1} e^{-t} t^{x+1} dt$$
$$< n^{-1} \int_0^\infty e^{-t} t^{x+1} dt \to 0,$$
as $n \to \infty$, since the last integral converges.

* This analysis is a modification of that given by Schlömilch, *Compendium der höheren Analysis*, II. p. 243. A simple method of obtaining a less precise inequality (which is sufficient for the object required) is given by Bromwich, *Infinite Series*, p. 459.

Consequently $\Gamma_1(z)=\Gamma(z)$ when the integral, by which $\Gamma_1(z)$ is defined, converges; that is to say that, when the real part of z is positive,

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt.$$

And so, when the real part of z is positive, $\Gamma(z)$ may be defined either by this integral or by the Weierstrassian product.

Example 1. Prove that, when $R(z)$ is positive,

$$\Gamma(z)=\int_0^1\left(\log\frac{1}{x}\right)^{z-1}dx.$$

Example 2. Prove that, if $R(z)>0$ and $R(s)>0$,

$$\int_0^\infty e^{-sx}x^{s-1}dx=\frac{\Gamma(s)}{z^s}.$$

Example 3. Prove that, if $R(z)>0$ and $R(s)>1$,

$$\frac{1}{(z+1)^s}+\frac{1}{(z+2)^s}+\frac{1}{(z+3)^s}+\ldots=\frac{1}{\Gamma(s)}\int_0^\infty\frac{e^{-xz}x^{s-1}dx}{e^x-1}$$

Example 4. From § 12·1 example 2, by using the inequality

$$0\leqslant e^{-t}-\left(1-\frac{t}{n}\right)^n\leqslant t^2e^{-t}/n,$$

deduce that

$$\gamma=\int_0^1\frac{1-e^{-t}-e^{-1/t}}{t}dt.$$

12·21. *Extension of the infinite integral to the case in which the argument of the Gamma-function is negative.*

The formula of the last article is no longer applicable when the real part of z is negative. Cauchy* and Saalschütz† have shewn, however, that, for negative arguments, an analogous theorem exists. This can be obtained in the following way.

Consider the function

$$\Gamma_2(z)=\int_0^\infty t^{z-1}\left(e^{-t}-1+t-\frac{t^2}{2!}+\ldots+(-)^{k+1}\frac{t^k}{k!}\right)dt,$$

where k is the integer so chosen that $-k>x>-k-1$, x being the real part of z.

By partial integration we have, when $z<-1$,

$$\Gamma_2(z)=\left[\frac{t^z}{z}\left(e^{-t}-1+t-\frac{t^2}{2!}+\ldots+(-)^{k+1}\frac{t^k}{k!}\right)\right]_0^\infty$$
$$+\frac{1}{z}\int_0^\infty t^z\left(e^{-t}-1+t-\ldots+(-)^k\frac{t^{k-1}}{(k-1)!}\right)dt.$$

The integrated part tends to zero at each limit, since $x+k$ is negative and $x+k+1$ is positive: so we have

$$\Gamma_2(z)=\frac{1}{z}\Gamma_2(z+1).$$

The same proof applies when x lies between 0 and -1, and leads to the result

$$\Gamma(z+1)=z\Gamma_2(z)\qquad(0>x>-1).$$

The last equation shews that, between the values 0 and -1 of x,

$$\Gamma_2(z)=\Gamma(z).$$

* *Exercices de Math.* II. (1827), pp. 91–92.

† *Zeitschrift für Math. und Phys.* XXXII. (1887), XXXIII. (1888).

The preceding equation then shews that $\Gamma_2(z)$ is the same as $\Gamma(z)$ for all negative values of $R(z)$ less than -1. Thus, for all negative values of $R(z)$, we have the result of Cauchy and Saalschütz

$$\Gamma(z)=\int_0^\infty t^{z-1}\left(e^{-t}-1+t-\frac{t^2}{2!}+\ldots+(-)^{k+1}\frac{t^k}{k!}\right)dt,$$

where k is the integer next less than $-R(z)$.

Example. If a function $P(\mu)$ be such that for positive values of μ we have

$$P(\mu)=\int_0^1 x^{\mu-1}e^{-x}dx,$$

and if for negative values of μ we define $P_1(\mu)$ by the equation

$$P_1(\mu)=\int_0^1 x^{\mu-1}\left(e^{-x}-1+x-\ldots+(-)^{k+1}\frac{x^k}{k!}\right)dx,$$

where k is the integer next less than $-\mu$, shew that

$$P_1(\mu)=P(\mu)-\frac{1}{\mu}+\frac{1}{1!(\mu+1)}-\ldots+(-)^{k-1}\frac{1}{k!(\mu+k)}.$$ (Saalschütz.)

12·22. *Hankel's expression of* $\Gamma(z)$ *as a contour integral.*

The integrals obtained for $\Gamma(z)$ in §§ 12·2, 12·21 are members of a large class of definite integrals by which the Gamma-function can be defined. The most general integral of the class in question is due to Hankel*; this integral will now be investigated.

Let D be a contour which starts from a point ρ on the real axis, encircles the origin once counter-clockwise and returns to ρ.

Consider $\int_D(-t)^{z-1}e^{-t}dt$, when the real part of z is positive and z is not an integer.

The many-valued function $(-t)^{z-1}$ is to be made definite by the convention that $(-t)^{z-1}=e^{(z-1)\log(-t)}$ and $\log(-t)$ is purely real when t is on the negative part of the real axis, so that, on D, $-\pi\leqslant\arg(-t)\leqslant\pi$.

The integrand is not analytic inside D, but, by § 5·2 corollary 1, the path of integration may be deformed (without affecting the value of the integral) into the path of integration which starts from ρ, proceeds along the real axis to δ, describes a circle of radius δ counter-clockwise round the origin and returns to ρ along the real axis.

On the real axis in the first part of this new path we have $\arg(-t)=-\pi$, so that $(-t)^{z-1}=e^{-i\pi(z-1)}t^{z-1}$ (where $\log t$ is purely real); and on the last part of the new path $(-t)^{z-1}=e^{i\pi(z-1)}t^{z-1}$.

On the circle we write $-t=\delta e^{i\theta}$; then we get

$$\int_D(-t)^{z-1}e^{-t}dt=\int_\rho^\delta e^{-i\pi(z-1)}t^{z-1}e^{-t}dt+\int_{-\pi}^{\pi}(\delta e^{i\theta})^{z-1}e^{\delta(\cos\theta+i\sin\theta)}\delta e^{i\theta}i\,d\theta$$
$$+\int_\delta^\rho e^{i\pi(z-1)}t^{z-1}e^{-t}dt$$
$$=-2i\sin(\pi z)\int_\delta^\rho t^{z-1}e^{-t}dt+i\delta^z\int_{-\pi}^{\pi}e^{iz\theta+\delta(\cos\theta+i\sin\theta)}d\theta.$$

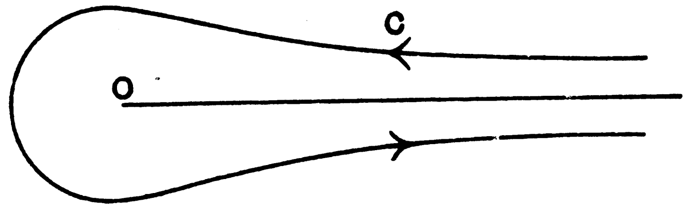
* *Zeitschrift für Math. und Phys.* IX. (1864), p. 7.

This is true for all positive values of $\delta \leqslant \rho$; now make $\delta \to 0$; then $\delta^z \to 0$ and $\int_{-\pi}^{\pi} e^{iz\theta+\delta(\cos\theta+i\sin\theta)}\,d\theta \to \int_{-\pi}^{\pi} e^{iz\theta}\,d\theta$ since the integrand tends to its limit uniformly.

We consequently infer that

$$\int_D (-t)^{z-1}e^{-t}\,dt = -2i\sin(\pi z)\int_0^\rho t^{z-1}e^{-t}\,dt.$$

This is true for all positive values of ρ; make $\rho \to \infty$, and let C be the limit of the contour D.



Then $$\int_C (-t)^{z-1}e^{-t}\,dt = -2i\sin(\pi z)\int_0^\infty t^{z-1}e^{-t}\,dt.$$

Therefore $$\Gamma(z) = -\frac{1}{2i\sin\pi z}\int_C (-t)^{z-1}e^{-t}\,dt.$$

Now, since the contour C does not pass through the point $t=0$, there is no need longer to stipulate that the real part of z is positive; and $\int_C (-t)^{z-1}e^{-t}\,dt$ is a one-valued analytic function of z for all values of z. *Hence, by* § 5·5, *the equation, just proved when the real part of z is positive, persists for* all *values of z with the exception of the values* $0, \pm 1, \pm 2, \ldots$.

Consequently, for all except integer values of z,

$$\Gamma(z) = -\frac{1}{2i\sin\pi z}\int_C (-t)^{z-1}e^{-t}\,dt.$$

This is Hankel's formula; if we write $1-z$ for z and make use of § 12·14, we get the further result that

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_C (-t)^{-z}e^{-t}\,dt.$$

We shall write $\int_\infty^{(0+)}$ for $\int_C$, meaning thereby that the path of integration starts at 'infinity' on the real axis, encircles the origin in the positive direction and returns to the starting point.

Example 1. Shew that, if the real part of z be positive and if a be any positive constant, $\int(-t)^{-z}e^{-t}\,dt$ tends to zero as $\rho \to \infty$, when the path of integration is either of the quadrants of circles of radius $\rho+a$ with centres at $-a$, the end points of one quadrant being ρ and $-a+i(\rho+a)$, and of the other ρ and $-a-i(\rho+a)$.

Deduce that $$\lim_{\rho\to\infty}\int_{-a+i\rho}^{-a-i\rho}(-t)^{-z}e^{-t}\,dt=\lim_{\rho\to\infty}\int_C(-t)^{-z}e^{-t}\,dt,$$
and hence, by writing $t=-a-iu$, shew that
$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{a+iu}(a+iu)^{-z}\,du.$$

[This formula was given by Laplace, *Théorie Analytique des Probabilités* (1812), p. 134, and it is substantially equivalent to Hankel's formula involving a contour integral.]

Example 2. By taking $a=1$, and putting $t=-1+i\tan\theta$ in example 1, shew that
$$\frac{1}{\Gamma(z)}=\frac{e}{\pi}\int_0^{\frac{1}{2}\pi}\cos(\tan\theta-z\theta)\cos^{z-2}\theta\,d\theta.$$

Example 3. By taking as contour of integration a parabola whose focus is the origin, shew that, if $a>0$, then
$$\Gamma(z)=\frac{2a^z e^a}{\sin\pi z}\int_0^{\infty}e^{-at^2}(1+t^2)^{z-\frac{1}{2}}\cos\{2at+(2z-1)\arc\tan t\}\,dt.$$
(Bourguet, *Acta Math.* I.)

Example 4. Investigate the values of x for which the integral
$$\frac{2}{\pi}\int_0^{\infty}t^{x-1}\sin t\,dt$$
converges; for such values of x express it in terms of Gamma-functions, and thence shew that it is equal to
$$e^{-\gamma x}\prod_{n=1}^{\infty}\left\{\left(1-\frac{x}{2n}\right)e^{x/(2n)}\right\}\Big/\prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{2n-1}\right)e^{-x/(2n-1)}\right\}.$$
(St John's, 1902.)

Example 5. Prove that $\int_0^{\infty}(\log t)^m\frac{\sin t}{t}\,dt$ converges when $m>0$, and, by means of example 4, evaluate it when $m=1$ and when $m=2$. (St John's, 1902.)

12·3. *Gauss' expression for the logarithmic derivate of the Gamma-function as an infinite integral**.

We shall now express the function $\frac{d}{dz}\log\Gamma(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ as an infinite integral when the real part of z is positive; the function in question is frequently written $\psi(z)$. We first need a new formula for γ.

Take the formula (§ 12·2 example 4)
$$\gamma=\int_0^1\frac{1-e^{-t}}{t}\,dt-\int_1^{\infty}\frac{e^{-t}}{t}\,dt=\lim_{\delta\to0}\left\{\int_\delta^1\frac{dt}{t}-\int_\delta^{\infty}\frac{e^{-t}}{t}\,dt\right\}=\lim_{\delta\to0}\left\{\int_\Delta^1\frac{dt}{t}-\int_\delta^{\infty}\frac{e^{-t}}{t}\,dt\right\},$$
where $\Delta=1-e^{-\delta}$, since $\quad\int_\Delta^\delta\frac{dt}{t}=\log\frac{\delta}{1-e^{-\delta}}\to0$ as $\delta\to0$.

Writing $t=1-e^{-u}$ in the first of these integrals and then replacing u by t we have
$$\gamma=\lim_{\delta\to0}\left\{\int_\delta^{\infty}\frac{e^{-t}}{1-e^{-t}}\,dt-\int_\delta^{\infty}\frac{e^{-t}}{t}\,dt\right\}=\int_0^{\infty}\left\{\frac{1}{1-e^{-t}}-\frac{1}{t}\right\}e^{-t}\,dt.$$
This is the formula for γ which was required.

* *Werke*, III. p. 159.

To get Gauss' formula, take the equation (§ 12·16)

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \lim_{n\to\infty} \sum_{m=1}^{n} \left(\frac{1}{m} - \frac{1}{z+m}\right),$$

and write

$$\frac{1}{z+m} = \int_0^\infty e^{-t(z+m)}\,dt\,;$$

this is permissible when $m = 0, 1, 2, \ldots$ if the real part of z is positive.

It follows that

$$\begin{aligned}\frac{\Gamma'(z)}{\Gamma(z)} &= -\gamma - \int_0^\infty e^{-zt}\,dt + \lim_{n\to\infty} \int_0^\infty \sum_{m=1}^{n} (e^{-mt} - e^{-(m+z)t})\,dt \\ &= -\gamma + \lim_{n\to\infty} \int_0^\infty \frac{e^{-t} - e^{-zt} - e^{-(n+1)t} + e^{-(z+n+1)t}}{1-e^{-t}}\,dt \\ &= \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt - \lim_{n\to\infty} \int_0^\infty \frac{1-e^{-zt}}{1-e^{-t}} e^{-(n+1)t}\,dt.\end{aligned}$$

Now, when $0 < t \leqslant 1$, $\left|\frac{1-e^{-zt}}{1-e^{-t}}\right|$ is a bounded function of t whose limit as $t\to 0$ is finite; and when $t \geqslant 1$,

$$\left|\frac{1-e^{-zt}}{1-e^{-t}}\right| < \frac{1+|e^{-zt}|}{1-e^{-1}} < \frac{2}{1-e^{-1}}.$$

Therefore we can find a number K independent of t such that, on the path of integration,

$$\left|\frac{1-e^{-zt}}{1-e^{-t}}\right| < K;$$

and so

$$\left|\int_0^\infty \frac{1-e^{-zt}}{1-e^{-t}} e^{(n+1)t}\,dt\right| < K \int_0^\infty e^{-(n+1)t}\,dt = K(n+1)^{-1} \to 0 \text{ as } n\to\infty.$$

We have thus proved the formula

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt,$$

which is Gauss' expression of $\psi(z)$ as an infinite integral. It may be remarked that this is the first integral which we have encountered connected with the Gamma-function in which the integrand is a single-valued function.

Writing $t = \log(1+x)$ in Gauss' result, we get, if $\Delta = e^\delta - 1$,

$$\begin{aligned}\frac{\Gamma'(z)}{\Gamma(z)} &= \lim_{\delta\to 0} \int_\delta^\infty \left\{\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right\} dt \\ &= \lim_{\delta\to 0} \left\{\int_\delta^\infty \frac{e^{-t}}{t}\,dt - \int_\Delta^\infty \frac{dx}{x(1+x)^z}\right\} \\ &= \lim_{\delta\to 0} \left\{\int_\Delta^\infty \frac{e^{-t}}{t}\,dt - \int_\Delta^\infty \frac{dx}{x(1+x)^z}\right\},\end{aligned}$$

since

$$0 < \int_\delta^\Delta \frac{e^{-t}}{t}\,dt < \int_\delta^\Delta \frac{dt}{t} = \log\frac{e^\delta - 1}{\delta} \to 0 \text{ as } \delta\to 0.$$

Hence

$$\frac{\Gamma'(z)}{\Gamma(z)} = \lim_{\Delta\to 0} \int_\Delta^\infty \left\{e^{-x} - \frac{1}{(1+x)^z}\right\} \frac{dx}{x},$$

so that

$$\Gamma'(z) = \Gamma(z) \int_0^\infty \left\{e^{-x} - \frac{1}{(1+x)^z}\right\} \frac{dx}{x},$$

an equation due to Dirichlet*.

* *Werke*, I. p. 275.

Example 1. Prove that, if the real part of z is positive,

$$\psi(z)=\int_0^1 \left\{\frac{1}{-\log t}-\frac{t^{z-1}}{1-t}\right\} dt. \qquad \text{(Gauss.)}$$

Example 2. Shew that $$\gamma=\int_0^\infty \{(1+t)^{-1}-e^{-t}\}\, t^{-1}\, dt. \qquad \text{(Dirichlet.)}$$

12·31. *Binet's first expression for* $\log\Gamma(z)$ *in terms of an infinite integral.*

Binet* has given two expressions for $\log\Gamma(z)$ which are of great importance as shewing the way in which $\log\Gamma(z)$ behaves as $|z|\to\infty$. To obtain the first of these expressions, we observe that, when the real part of z is positive,

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)}=\int_0^\infty \left\{\frac{e^{-t}}{t}-\frac{e^{-tz}}{e^t-1}\right\} dt,$$

writing $z+1$ for z in § 12·3.

Now, by § 6·222 example 6, we have

$$\log z=\int_0^\infty \frac{e^{-t}-e^{-tz}}{t}\, dt,$$

and so, since $$(2z)^{-1}=\int_0^\infty \frac{1}{2} e^{-tz}\, dt,$$

we have

$$\frac{d}{dz}\log\Gamma(z+1)=\frac{1}{2z}+\log z-\int_0^\infty \left\{\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right\} e^{-tz}\, dt.$$

The integrand in the last integral is continuous as $t\to 0$; and since $\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}$ is bounded as $t\to\infty$, it follows without difficulty that the integral converges uniformly when the real part of z is positive; we may consequently integrate from 1 to z under the sign of integration (§ 4·44) and we get†

$$\log\Gamma(z+1)=\left(z+\frac{1}{2}\right)\log z-z+1+\int_0^\infty \left\{\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right\}\frac{e^{-tz}-e^{-t}}{t}\, dt.$$

Since $\left\{\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right\}\frac{1}{t}$ is continuous as $t\to 0$ by § 7·2, and since

$$\log\Gamma(z+1)=\log z+\log\Gamma(z),$$

we have

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+1+\int_0^\infty \left\{\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right\}\frac{e^{-tz}}{t}\, dt$$
$$-\int_0^\infty \left\{\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right\}\frac{e^{-t}}{t}\, dt.$$

* *Journal de l'École Polytechnique*, XVI. (1839), pp. 123–143.

† $\operatorname{Log}\Gamma(z+1)$ means the sum of the principal values of the logarithms in the factors of the Weierstrassian product.

To evaluate the second of these integrals, let*

$$\int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-t}}{t}\, dt = I, \qquad \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-\frac{1}{2}t}}{t}\, dt = J;$$

so that, taking $z=\frac{1}{2}$ in the last expression for $\log \Gamma(z)$, we get

$$\tfrac{1}{2}\log \pi = \tfrac{1}{2} + J - I.$$

Also, since $I = \int_0^\infty \left(\frac{1}{2} - \frac{2}{t} + \frac{1}{e^{\frac{1}{2}t} - 1}\right) \frac{e^{-\frac{1}{2}t}}{t}\, dt$, we have

$$\begin{aligned} J - I &= \int_0^\infty \left(\frac{1}{t} - \frac{e^{\frac{1}{2}t}}{e^t - 1}\right) \frac{e^{-\frac{1}{2}t}\, dt}{t} \\ &= \int_0^\infty \left(\frac{e^{-\frac{1}{2}t}}{t} - \frac{1}{e^t - 1}\right) \frac{dt}{t}. \end{aligned}$$

And so

$$\begin{aligned} J &= \int_0^\infty \left\{\frac{e^{-\frac{1}{2}t}}{t} - \frac{1}{e^t - 1} + \tfrac{1}{2}e^{-t} - \frac{e^{-t}}{t} + \frac{e^{-t}}{e^t - 1}\right\} \frac{dt}{t} \\ &= \int_0^\infty \left\{\frac{e^{-\frac{1}{2}t} - e^{-t}}{t} - \tfrac{1}{2}e^{-t}\right\} \frac{dt}{t} \\ &= \int_0^\infty \left\{-\frac{d}{dt}\left(\frac{e^{-\frac{1}{2}t} - e^{-t}}{t}\right) - \frac{\frac{1}{2}e^{-\frac{1}{2}t} - e^{-t}}{t} - \frac{e^{-t}}{2t}\right\} dt \\ &= \left[-\frac{e^{-\frac{1}{2}t} - e^{-t}}{t}\right]_0^\infty + \tfrac{1}{2}\int_0^\infty \frac{e^{-t} - e^{-\frac{1}{2}t}}{t}\, dt \\ &= \tfrac{1}{2} + \tfrac{1}{2}\log \tfrac{1}{2}. \end{aligned}$$

Consequently

$$I = 1 - \tfrac{1}{2}\log(2\pi).$$

We therefore have Binet's result that, when the real part of z is positive,

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tz}}{t}\, dt.$$

If $z = x + iy$, we see that, if the upper bound of $\left|\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{1}{t}\right|$ for real values of t is K, then

$$\begin{aligned} \left|\log \Gamma(z) - \left(z - \frac{1}{2}\right)\log z + z - \frac{1}{2}\log(2\pi)\right| &< K\int_0^\infty e^{-tx}\, dt \\ &= Kx^{-1}, \end{aligned}$$

so that, when x is large, the terms $\left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi)$ furnish an approximate expression for $\log \Gamma(z)$.

Example 1. Prove that, when $R(z) > 0$,

$$\log \Gamma(z) = \int_0^\infty \left\{\frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z-1)\, e^{-t}\right\} \frac{dt}{t}. \qquad \text{(Malmstén.)}$$

Example 2. Prove that, when $R(z) > 0$,

$$\log \Gamma(z) = \int_0^\infty \left\{(z-1)\, e^{-t} + \frac{(1+t)^{-z} - (1+t)^{-1}}{\log(1+t)}\right\} \frac{dt}{t}. \qquad \text{(Féaux.)}$$

* This artifice is due to Pringsheim, *Math. Ann.* XXXI. (1888), p. 473.

Example 3. From the formula of § 12·14, shew that, if $0<x<1$,

$$2\log\Gamma(x)-\log\pi+\log\sin\pi x=\int_0^\infty\left\{\frac{\sinh(\frac{1}{2}-x)t}{\sinh\frac{1}{2}t}-(1-2x)e^{-t}\right\}\frac{dt}{t}.$$

(Kummer.)

Example 4. By expanding $\sinh(\frac{1}{2}-x)t$ and $1-2x$ in Fourier sine series, shew from example 3 that, if $0<x<1$,

$$\log\Gamma(x)=\tfrac{1}{2}\log\pi-\tfrac{1}{2}\log\sin\pi x+2\sum_{n=1}^{\infty}a_n\sin 2n\pi x,$$

where
$$a_n=\int_0^\infty\left\{\frac{2n\pi}{t^2+4n^2\pi^2}-\frac{e^{-t}}{2n\pi}\right\}\frac{dt}{t}.$$

Deduce from example 2 of § 12·3 that

$$a_n=\frac{1}{2n\pi}(\gamma+\log 2\pi+\log n).$$

(Kummer, *Journal für Math.* xxxv. (1847), p. 1.)

12·32. *Binet's second expression for* $\log\Gamma(z)$ *in terms of an infinite integral.*

Consider the application of example 7 of Chapter VII (p. 145) to the equation (§ 12·16)

$$\frac{d^2}{dz^2}\log\Gamma(z)=\sum_{n=0}^{\infty}\frac{1}{(z+n)^2}.$$

The conditions there stated as sufficient for the transformation of a series into integrals are obviously satisfied by the function $\phi(\zeta)=\dfrac{1}{(z+\zeta)^2}$, if the real part of z be positive; and we have

$$\frac{d^2}{dz^2}\log\Gamma(z)=\frac{1}{2z^2}+\int_0^\infty\frac{d\xi}{(z+\xi)^2}-2\int_0^\infty\frac{q(t,z)}{e^{2\pi t}-1}dt+2\lim_{n\to\infty}\int_0^\infty\frac{q(t,z+n)}{e^{2\pi t}-1}dt,$$

where
$$2iq(t)=\frac{1}{(z+it)^2}-\frac{1}{(z-it)^2}.$$

Since $|q(t,z+n)|$ is easily seen to be less than K_1t/n, where K_1 is independent of t and n, it follows that the limit of the last integral is zero.

Hence
$$\frac{d^2}{dz^2}\log\Gamma(z)=\frac{1}{2z^2}+\frac{1}{z}+\int_0^\infty\frac{4tz}{(z^2+t^2)^2}\frac{dt}{e^{2\pi t}-1}.$$

Since $\left|\dfrac{2z}{z^2+t^2}\right|$ does not exceed K (where K depends only on δ) when the real part of z exceeds δ, the integral converges uniformly and we may integrate under the integral sign (§ 4·44) from 1 to z.

We get

$$\frac{d}{dz}\log\Gamma(z)=-\frac{1}{2z}+\log z+C-2\int_0^\infty\frac{t\,dt}{(z^2+t^2)(e^{2\pi t}-1)},$$

where C is a constant. Integrating again,

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z+(C-1)z+C'+2\int_0^\infty\frac{\arctan(t/z)}{e^{2\pi t}-1}dt,$$

where C' is a constant.

Now, if z is real, $\quad 0 \leqslant \arctan t/z \leqslant t/z,$

and so

$$\left|\log \Gamma(z) - \left(z - \frac{1}{2}\right)\log z - (C-1)z - C'\right| < \frac{2}{z}\int_0^\infty \frac{t}{e^{2\pi t}-1}\,dt.$$

But it has been shewn in § 12·31 that

$$\left|\log \Gamma(z) - \left(z - \frac{1}{2}\right)\log z + z - \frac{1}{2}\log(2\pi)\right| \to 0,$$

as $z \to \infty$ through real values. Comparing these results we see that $C = 0$, $C' = \frac{1}{2}\log(2\pi)$.

Hence for all values of z whose real part is positive,

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + 2\int_0^\infty \frac{\arctan(t/z)}{e^{2\pi t}-1}\,dt,$$

where $\arctan u$ is defined by the equation

$$\arctan u = \int_0^u \frac{dt}{1+t^2},$$

in which the path of integration is a straight line.

This is Binet's second expression for $\log \Gamma(z)$.

Example. Justify differentiating with regard to z under the sign of integration, so as to get the equation

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - 2\int_0^\infty \frac{t\,dt}{(t^2+z^2)(e^{2\pi t}-1)}.$$

12·33. THE ASYMPTOTIC EXPANSION OF THE LOGARITHM OF THE GAMMA-FUNCTION (STIRLING'S SERIES).

We can now obtain an expansion which represents the function $\log \Gamma(z)$ asymptotically (§ 8·2) for large values of $|z|$, and which is used in the calculation of the Gamma-function.

Let us assume that, if $z = x + iy$, then $x \geqslant \delta > 0$; and we have, by Binet's second formula,

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \phi(z),$$

where $$\phi(z) = 2\int_0^\infty \frac{\arctan(t/z)}{e^{2\pi t}-1}\,dt.$$

Now

$$\arctan(t/z) = \frac{t}{z} - \frac{1}{3}\frac{t^3}{z^3} + \frac{1}{5}\frac{t^5}{z^5} - \dots + \frac{(-)^{n-1}}{2n-1}\frac{t^{2n-1}}{z^{2n-1}} + \frac{(-)^n}{z^{n-1}}\int_0^t \frac{u^{2n}\,du}{u^2+z^2}.$$

Substituting and remembering (§ 7·2) that

$$\int_0^\infty \frac{t^{2n-1}\,dt}{e^{2\pi t}-1} = \frac{B_n}{4n},$$

where $B_1, B_2, \ldots$ are Bernoulli's numbers, we have

$$\phi(z) = \sum_{r=1}^{n} \frac{(-)^{r-1} B_r}{2r(2r-1)z^{2r-1}} + \frac{2(-)^n}{z^{2n-1}} \int_0^\infty \left\{\int_0^t \frac{u^{2n}\,du}{u^2+z^2}\right\} \frac{dt}{e^{2\pi t}-1}.$$

Let the upper bound* of $\left|\frac{z^2}{u^2+z^2}\right|$ for positive values of u be K_z.

Then

$$\left|\int_0^\infty \left\{\int_0^t \frac{u^{2n}\,du}{u^2+z^2}\right\} \frac{dt}{e^{2\pi t}-1}\right| \leqslant K_z |z|^{-2} \int_0^\infty \left\{\int_0^t u^{2n}\,du\right\} \frac{dt}{e^{2\pi t}-1}$$

$$\leqslant \frac{K_z B_{n+1}}{4(n+1)(2n+1)|z|^2}.$$

Hence

$$\left|\frac{2(-)^n}{z^{2n-1}} \int_0^\infty \left\{\int_0^t \frac{u^{2n}\,du}{u^2+z^2}\right\} \frac{dt}{e^{2\pi t}-1}\right| < \frac{K_z B_{n+1}}{2(n+1)(2n+1)|z|^{2n+1}},$$

and it is obvious that this tends to zero uniformly as $|z| \to \infty$ if $|\arg z| \leqslant \frac{1}{2}\pi - \Delta$, where $\frac{1}{4}\pi > \Delta > 0$, so that $K_z \leqslant \operatorname{cosec} 2\Delta$.

Also it is clear that if $|\arg z| \leqslant \frac{1}{4}\pi$ (so that $K_z = 1$) the error in taking the first n terms of the series

$$\sum_{r=1}^{\infty} \frac{(-)^{r-1} B_r}{2r(2r-1)} \frac{1}{z^{2r-1}}$$

as an approximation to $\phi(z)$ is numerically less than the $(n+1)$th term.

Since, if $|\arg z| \leqslant \frac{1}{2}\pi - \Delta$,

$$\left| z^{2n-1} \left\{\phi(z) - \sum_{r=1}^{n} \frac{(-)^{r-1} B_r}{2r(2r-1)}\right\}\right| < \operatorname{cosec}^2 2\Delta \cdot \frac{B_{n+1}}{2(n+1)(2n+1)} |z|^{-2}$$

$$\to 0,$$

as $z \to \infty$, it is clear that

$$\frac{B_1}{1.2.z} - \frac{B_2}{3.4.z^3} + \frac{B_3}{5.6.z^5} - \ldots$$

is the asymptotic expansion† (§ 8·2) of $\phi(z)$.

We see therefore that the series

$$\left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{r=1}^{\infty} \frac{(-)^{r-1} B_r}{2r(2r-1)z^{2r-1}}$$

is the asymptotic expansion of $\log \Gamma(z)$ when $|\arg z| \leqslant \frac{1}{2}\pi - \Delta$.

* K_z^{-2} is the lower bound of $\dfrac{\{u^2+(x^2-y^2)\}^2+4x^2y^2}{(x^2+y^2)^2}$ and is consequently equal to

$$\frac{4x^2y^2}{(x^2+y^2)^2} \text{ or } 1 \text{ as } x^2<y^2 \text{ or } x^2 \geqslant y^2.$$

† The development is asymptotic; for if it converged when $|z| \geqslant \rho$, by § 2·6 we could find K, such that $B_n < (2n-1)\,2nK\rho^{2n}$; and then the series $\sum_{n=1}^{\infty} \frac{(-)^{n-1} B_n t^{2n}}{(2n)!}$ would define an integral function; this is contrary to § 7·2.

This is generally known as *Stirling's series*. In § 13·6 it will be established over the extended range $|\arg z| \leqslant \pi - \Delta$.

In particular when z is positive ($= x$), we have

$$0 < 2\int_0^\infty \left\{\int_0^t \frac{u^{2n}\,du}{u^2+x^2}\right\} \frac{dt}{e^{2\pi t}-1} < \frac{B_{n+1}}{2(n+1)(2n+1)x^2}.$$

Hence, when $x > 0$, the value of $\phi(x)$ always lies between the sum of n terms and the sum of $n+1$ terms of the series for all values of n.

In particular $0 < \phi(x) < \dfrac{B_1}{1\,.\,2x}$, so that $\phi(x) = \dfrac{\theta}{12x}$ where $0 < \theta < 1$.

Hence $$\Gamma(x) = x^{x-\frac{1}{2}} e^{-x} (2\pi)^{\frac{1}{2}} e^{\theta/(12x)}.$$

Also, taking the exponential of Stirling's series, we get

$$\Gamma(x) = e^{-x} x^{x-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left\{1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + O\left(\frac{1}{x^5}\right)\right\}.$$

This is an *asymptotic formula for the Gamma-function*. In conjunction with the formula $\Gamma(x+1) = x\Gamma(x)$, it is very useful for the purpose of computing the numerical value of the function for real values of x.

Tables of the function $\log_{10}\Gamma(x)$, correct to 12 decimal places, for values of x between 1 and 2, were constructed in this way by Legendre, and published in his *Exercices de Calcul Intégral*, II. p. 85, in 1817, and his *Traité des fonctions elliptiques* (1826), p. 489.

It may be observed that $\Gamma(x)$ has one minimum for positive values of x, when $x = 1{\cdot}4616321\ldots$, the value of $\log_{10}\Gamma(x)$ then being $\bar{1}{\cdot}9472391\ldots$.

Example. Obtain the expansion, convergent when $R(z) > 0$,

$$\log_e \Gamma(z) = (z - \tfrac{1}{2}) \log_e z - z + \tfrac{1}{2}\log_e (2\pi) + J(z),$$

where

$$J(z) = \tfrac{1}{2}\left\{\frac{c_1}{z+1} + \frac{c_2}{2(z+1)(z+2)} + \frac{c_3}{3(z+1)(z+2)(z+3)} + \ldots\right\},$$

in which

$$c_1 = \tfrac{1}{6},\quad c_2 = \tfrac{1}{3},\quad c_3 = \tfrac{59}{60},\quad c_4 = \tfrac{227}{60},$$

and generally

$$c_n = \int_0^1 (x+1)(x+2)\ldots(x+n-1)(2x-1)\,x\,dx. \qquad \text{(Binet.)}$$

12·4. *The Eulerian Integral of the First Kind.*

The name *Eulerian Integral of the First Kind* was given by Legendre to the integral

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx,$$

which was first studied by Euler and Legendre*. In this integral, the real parts of p and q are supposed to be positive; and x^{p-1}, $(1-x)^{q-1}$ are to be understood to mean those values of $e^{(p-1)\log x}$ and $e^{(q-1)\log(1-x)}$ which correspond to the real determinations of the logarithms.

* Euler, *Nov. Comm. Petrop.* XVI. (1772); Legendre, *Exercices*, I. p. 221.

With these stipulations, it is easily seen that $B(p, q)$ exists, as a (possibly improper) integral (§ 4·5 example 2).

We have, on writing $(1-x)$ for x,

$$B(p, q) = B(q, p).$$

Also, integrating by parts,

$$\int_0^1 x^{p-1}(1-x)^q\,dx = \left[\frac{x^p(1-x)^q}{p}\right]_0^1 + \frac{q}{p}\int_0^1 x^p(1-x)^{q-1}\,dx,$$

so that

$$B(p, q+1) = \frac{q}{p} B(p+1, q).$$

Example 1. Shew that

$$B(p, q) = B(p+1, q) + B(p, q+1).$$

Example 2. Deduce from example 1 that

$$B(p, q+1) = \frac{q}{p+q} B(p, q).$$

Example 3. Prove that if n is a positive integer,

$$B(p, n+1) = \frac{1 \,.\, 2 \ldots n}{p(p+1)\ldots(p+n)}.$$

Example 4. Prove that

$$B(x, y) = \int_0^\infty \frac{a^{x-1}}{(1+a)^{x+y}}\,da.$$

Example 5. Prove that

$$\Gamma(z) = \lim_{n\to\infty} n^z B(z, n).$$

12·41. *Expression of the Eulerian Integral of the First Kind in terms of the Gamma-function.*

We shall now establish the important theorem that

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

First let the real parts of m and n exceed $\frac{1}{2}$; then

$$\Gamma(m)\,\Gamma(n) = \int_0^\infty e^{-x} x^{m-1}\,dx \times \int_0^\infty e^{-y} y^{n-1}\,dy.$$

On writing x^2 for x, and y^2 for y, this gives

$$\Gamma(m)\,\Gamma(n) = 4 \lim_{R\to\infty} \int_0^R e^{-x^2} x^{2m-1}\,dx \times \int_0^R e^{-y^2} y^{2n-1}\,dy$$

$$= 4 \lim_{R\to\infty} \int_0^R \int_0^R e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy.$$

Now, for the values of m and n under consideration, the integrand is continuous over the range of integration, and so the integral may be considered as a double integral taken over a square S_R. Calling the integrand

$f(x, y)$, and calling Q_R the quadrant with centre at the origin and radius R, we have, if T_R be the part of S_R outside Q_R,

$$\left|\iint_{S_R} f(x, y)\, dx\, dy - \iint_{Q_R} f(x, y)\, dx\, dy\right|$$

$$= \left|\iint_{T_R} f(x, y)\, dx\, dy\right|$$

$$\leqslant \iint_{T_R} |f(x, y)|\, dx\, dy$$

$$\leqslant \iint_{S_R} |f(x, y)|\, dx\, dy - \iint_{S_{\frac{1}{2}R}} |f(x, y)\, dx\, dy|$$

$$\to 0 \text{ as } R \to \infty,$$

since $\iint_{S_R} |f(x, y)|\, dx\, dy$ converges to a limit, namely

$$2\int_0^\infty e^{-x^2} |x^{2m-1}|\, dx \times 2\int_0^\infty e^{-y} \;|y^{2n-1}|\, dy.$$

Therefore

$$\lim_{R\to\infty} \iint_{S_R} f(x, y)\, dx\, dy = \lim_{R\to\infty} \iint_{Q_R} f(x, y)\, dx\, dy.$$

Changing to polar* coordinates ($x = r\cos\theta$, $y = r\sin\theta$), we have

$$\iint_{Q_R} f(x, y)\, dx\, dy = \int_0^R \int_0^{\frac{1}{2}\pi} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\, dr\, d\theta.$$

Hence

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr \int_0^{\frac{1}{2}\pi} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

$$= 2\Gamma(m+n) \int_0^{\frac{1}{2}\pi} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta.$$

Writing $\cos^2\theta = u$ we at once get

$$\Gamma(m)\,\Gamma(n) = \Gamma(m+n)\,.\,B(m, n).$$

This has only been proved when the real parts of m and n exceed $\frac{1}{2}$; but it can obviously be deduced when these are less than $\frac{1}{2}$ by § 12·4 example 2.

This result, discovered by Euler, connects the Eulerian Integral of the First Kind with the Gamma-function.

Example 1. Shew that

$$\int_{-1}^1 (1+x)^{p-1} (1-x)^{q-1}\, dx = 2^{p+q-1} \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}.$$

* It is easily proved by the methods of § 4·11 that the areas $A_{m,\mu}$ of § 4·3 need not be rectangles provided only that their greatest diameters can be made arbitrarily small by taking the number of areas sufficiently large; so the areas may be taken to be the regions bounded by radii vectores and circular arcs.

Example 2. Shew that, if

$$f(x, y) = \frac{1}{x} - y\frac{1}{x+1} + \frac{y(y-1)}{2!}\frac{1}{x+2} - \frac{y(y-1)(y-2)}{3!}\frac{1}{x+3} + \dots,$$

then

$$f(x, y) = f(y+1, x-1),$$

where x and y have such values that the series are convergent. (Jesus, 1901.)

Example 3. Prove that

$$\int_0^1 \int_0^1 f(xy)(1-x)^{\mu-1} y^\mu (1-y)^{\nu-1} dx\, dy = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \int_0^1 f(z)(1-z)^{\mu+\nu-1} dz.$$

(Math. Trip. 1894.)

12·42. *Evaluation of trigonometrical integrals in terms of the Gamma-function.*

We can now evaluate the integral $\int_0^{\frac{1}{2}\pi} \cos^{m-1} x \sin^{n-1} x\, dx$, where m and n are not restricted to be integers, but have their real parts positive.

For, writing $\cos^2 x = t$, we have, as in § 12·41,

$$\int_0^{\frac{1}{2}\pi} \cos^{m-1} x \sin^{n-1} x\, dx = \frac{1}{2} \frac{\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}m + \frac{1}{2}n)}.$$

The well-known elementary formulae for the cases in which m and n are integers can be at once derived from this result.

Example. Prove that, when $|k| < 1$,

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^m \theta \sin^n \theta\, d\theta}{(1-k\sin^2\theta)^{\frac{1}{2}}} = \frac{\Gamma(\frac{1}{2}m+\frac{1}{2})\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2})\sqrt{\pi}} \int_0^{\frac{1}{2}\pi} \frac{\cos^{m+n}\theta\, d\theta}{(1-k\sin^2\theta)^{\frac{1}{2}n+\frac{1}{2}}}.$$

(Trinity, 1898.)

12·43. *Pochhammer's* extension of the Eulerian Integral of the First Kind.*

We have seen in § 12·22 that it is possible to replace the second Eulerian integral for $\Gamma(z)$ by a contour integral which converges for all values of z. A similar process has been carried out by Pochhammer for Eulerian integrals of the first kind.

Let P be any point on the real axis between 0 and 1; consider the integral

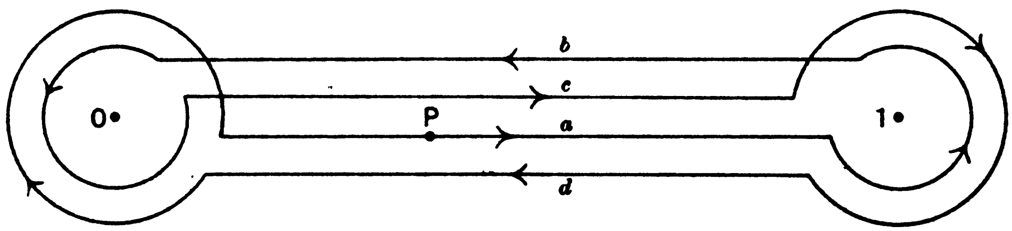
$$e^{-\pi i(\alpha+\beta)} \int_P^{(1+,\,0+,\,1-,\,0-)} t^{\alpha-1}(1-t)^{\beta-1} dt = \epsilon(\alpha, \beta).$$

The notation employed is that introduced at the end of § 12·22 and means that the path of integration starts from P, encircles the point 1 in the positive (counter-clockwise) direction and returns to P, then encircles the origin in the positive direction and returns to P, and so on.

* *Math. Ann.* xxxv. (1890), p. 495. The use of the double circuit integrals of this section seems to be due to Jordan, *Cours d'Analyse*, III. (1887).

At the starting-point the arguments of t and $1-t$ are both zero; after the circuit $(1+)$ they are 0 and 2π; after the circuit $(0+)$ they are 2π and 2π; after the circuit $(1-)$ they are 2π and 0 and after the circuit $(0-)$ they are both zero, so that the final value of the integrand is the same as the initial value.

It is easily seen that, since the path of integration may be deformed in any way so long as it does not pass over the branch points 0, 1 of the integrand, the path may be taken to be that shewn in the figure, wherein the four parallel lines are supposed to coincide with the real axis.



If the real parts of α and β are positive the integrals round the circles tend to zero as the radii of the circles tend to zero*; the integrands on the paths marked a, b, c, d are

$$t^{\alpha-1}(1-t)^{\beta-1}, \quad t^{\alpha-1}(1-t)^{\beta-1}e^{2\pi i(\beta-1)},$$
$$t^{\alpha-1}e^{2\pi i(\alpha-1)}(1-t)^{\beta-1}e^{2\pi i(\beta-1)}, \quad t^{\alpha-1}e^{2\pi i(\alpha-1)}(1-t)^{\beta-1}$$

respectively, the arguments of t and $1-t$ *now being zero in each case.*

Hence we may write $\epsilon(\alpha, \beta)$ as the sum of four (possibly improper) integrals, thus:

$$\epsilon(\alpha,\beta) = e^{-\pi i(\alpha+\beta)}\left[\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt + \int_1^0 t^{\alpha-1}(1-t)^{\beta-1}e^{2\pi i\beta}\,dt \right.$$
$$\left. + \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{2\pi i(\alpha+\beta)}\,dt + \int_1^0 t^{\alpha-1}(1-t)^{\beta-1}e^{2\pi i\alpha}\,dt\right].$$

Hence

$$\epsilon(\alpha,\beta) = e^{-\pi i(\alpha+\beta)}(1-e^{2\pi i\alpha})(1-e^{2\pi i\beta})\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$
$$= -4\sin(\alpha\pi)\sin(\beta\pi)\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$
$$= \frac{-4\pi^2}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta)}.$$

Now $\epsilon(\alpha, \beta)$ and this last expression are analytic functions of α and of β for *all* values of α and β. So, by the theory of analytic continuation, this equality, proved when the real parts of α and β are positive, holds for all values of α and β. *Hence for all values of α and β we have proved that*

$$\epsilon(\alpha,\beta) = \frac{-4\pi^2}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta)}.$$

* The reader ought to have no difficulty in proving this.

12·5. *Dirichlet's integral**.

We shall now shew how the repeated integral

$$I = \iint \dots \int f(t_1 + t_2 + \dots + t_n)\, t_1^{\alpha_1 - 1} t_2^{\alpha_2 - 1} \dots t_n^{\alpha_n - 1} dt_1 dt_2 \dots dt_n$$

may be reduced to a simple integral, where f is continuous, $\alpha_r > 0$ $(r = 1, 2, \dots n)$ and the integration is extended over all positive values of the variables such that $t_1 + t_2 + \dots + t_n \leqslant 1$.

To simplify $$\int_0^{1-\lambda} \int_0^{1-\lambda-T} f(t + T + \lambda)\, t^{\alpha-1} T^{\beta-1} dt\, dT$$

(where we have written t, T, α, β for $t_1, t_2, \alpha_1, \alpha_2$ and λ for $t_3 + t_4 + \dots + t_n$), put $t = T(1 - v)/v$; the integral becomes (if $\lambda \neq 0$)

$$\int_0^{1-\lambda} \int_{T/(1-\lambda)}^{1} f(\lambda + T/v)\,(1 - v)^{\alpha-1} v^{-\alpha-1} T^{\alpha+\beta-1} dv\, dT.$$

Changing the order of integration (§ 4·51), the integral becomes

$$\int_0^1 \int_0^{(1-\lambda)v} f(\lambda + T/v)\,(1 - v)^{\alpha-1} v^{-\alpha-1} T^{\alpha+\beta-1} dT\, dv.$$

Putting $T = v\tau_2$, the integral becomes

$$\int_0^1 \int_0^{1-\lambda} f(\lambda + \tau_2)\,(1 - v)^{\alpha-1} v^{\beta-1} \tau_2^{\alpha+\beta-1} d\tau_2\, dv$$
$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_0^{1-\lambda} f(\lambda + \tau_2)\, \tau_2^{\alpha+\beta-1} d\tau_2.$$

Hence

$$I = \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} \iint \dots \int f(\tau_2 + t_3 + \dots + t_n)\, \tau_2^{\alpha_1+\alpha_2-1} t_3^{\alpha_3-1} \dots t_n^{\alpha_n-1} d\tau_2 dt_3 \dots dt_n,$$

the integration being extended over all positive values of the variables such that $\tau_2 + t_3 + \dots + t_n \leqslant 1$.

Continually reducing in this way we get

$$I = \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2) \dots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \alpha_2 + \dots + \alpha_n)} \int_0^1 f(\tau)\, \tau^{\Sigma\alpha - 1} d\tau,$$

which is Dirichlet's result.

Example 1. Reduce

$$\iiint f\left\{\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta + \left(\frac{z}{c}\right)^\gamma\right\} x^{p-1} y^{q-1} z^{r-1} dx\, dy\, dz$$

to a simple integral; the range of integration being extended over all positive values of the variables such that

$$\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta + \left(\frac{z}{c}\right)^\gamma \leqslant 1,$$

it being assumed that $a, b, c, \alpha, \beta, \gamma, p, q, r$ are positive. (Dirichlet.)

* *Werke*, I. pp. 375, 391.

Example 2. Evaluate
$$\iint x^p y^q \, dx\, dy,$$
m and n being positive and
$$x \geqslant 0, \quad y \geqslant 0, \quad x^m + y^n \leqslant 1. \qquad \text{(Pembroke, 1907.)}$$

Example 3. Shew that the moment of inertia of a homogeneous ellipsoid of unit density, taken about the axis of z, is
$$\tfrac{4}{15}(a^2+b^2)\,\pi abc,$$
where a, b, c are the semi-axes.

Example 4. Shew that the area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}$ is $\frac{3}{8}\pi l^2$.

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MISCELLANEOUS EXAMPLES.

1. Shew that
$$(1-z)\left(1+\frac{z}{2}\right)\left(1-\frac{z}{3}\right)\left(1+\frac{z}{4}\right)\dots = \frac{\pi^{\frac{1}{2}}}{\Gamma(1+\frac{1}{2}z)\,\Gamma(\frac{1}{2}-\frac{1}{2}z)}.$$
(Trinity, 1897.)

2. Shew that
$$\lim_{n\to\infty} \frac{1}{1+x}\,\frac{1}{1+\frac{1}{2}x}\,\frac{1}{1+\frac{1}{3}x}\cdots\frac{1}{1+\frac{1}{n}x}\,n^x = \Gamma(x+1).$$
(Trinity, 1885.)

3. Prove that
$$\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = 2\log 2.$$
(Jesus, 1903.)

4. Shew that
$$\frac{\{\Gamma(\frac{1}{4})\}^4}{16\pi^2} = \frac{3^2}{3^2-1}\cdot\frac{5^2-1}{5^2}\cdot\frac{7^2}{7^2-1}\cdot\frac{9^2-1}{9^2}\cdot\frac{11^2}{11^2-1}\cdots.$$
(Trinity, 1891.)

5. Shew that
$$\prod_{n=0}^{\infty}\left\{\frac{(n-a)(n+\beta+\gamma)}{(n+\beta)(n+\gamma)}\left(1+\frac{a}{n+1}\right)\right\} = -\frac{1}{\pi}\sin(a\pi)\,B(\beta,\gamma).$$
(Trinity, 1905.)

6. Shew that
$$\prod_{r=1}^{8}\Gamma\left(\frac{r}{3}\right) = \frac{640}{3^6}\left(\frac{\pi}{\sqrt{3}}\right)^3.$$
(Peterhouse, 1906.)

7. Shew that, if $z = i\zeta$ where ζ is real, then
$$|\Gamma(z)| = \sqrt{\left(\frac{\pi}{\zeta \sinh \pi\zeta}\right)}.$$
(Trinity, 1904.)

8. When x is positive, shew that†
$$\frac{\Gamma(x)\,\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})} = \sum_{n=0}^{\infty}\frac{2n!}{2^{2n}\,.\,n!\,n!}\,\frac{1}{x+n}.$$
(Math. Trip. 1897.)

* This work contains a complete bibliography.

† This and some other examples are most easily proved by the result of § 14·11.

9. If a is positive, shew that

$$\frac{\Gamma(z)\Gamma(a+1)}{\Gamma(z+a)}=\sum_{n=0}^{\infty}\frac{(-)^n a(a-1)(a-2)\dots(a-n)}{n!}\frac{1}{z+n}.$$

10. If $x>0$ and

$$P(x)=\int_0^1 e^{-t}t^{x-1}\,dt,$$

shew that

$$P(x)=\frac{1}{x}-\frac{1}{1!}\frac{1}{x+1}+\frac{1}{2!}\frac{1}{x+2}-\frac{1}{3!}\frac{1}{x+3}+\dots,$$

and

$$P(x+1)=xP(x)-e^{-1}.$$

11. Shew that if $\lambda>0$, $x>0$, $-\frac{1}{2}\pi<a<\frac{1}{2}\pi$, then

$$\int_0^\infty t^{x-1}e^{-\lambda t\cos a}\cos(\lambda t\sin a)\,dt=\lambda^{-x}\Gamma(x)\cos ax,$$

$$\int_0^\infty t^{x-1}e^{-\lambda t\cos a}\sin(\lambda t\sin a)\,dt=\lambda^{-x}\Gamma(x)\sin ax.$$ (Euler.)

12. Prove that, if $b>0$, then, when $0<z<2$,

$$\int_0^\infty\frac{\sin bx}{x^z}\,dx=\tfrac{1}{2}\pi b^{z-1}\operatorname{cosec}(\tfrac{1}{2}\pi z)/\Gamma(z),$$

and, when $0<z<1$,

$$\int_0^\infty\frac{\cos bx}{x^z}\,dx=\tfrac{1}{2}\pi b^{z-1}\sec(\tfrac{1}{2}\pi z)/\Gamma(z).$$ (Euler.)

13. If $0<n<1$, prove that

$$\int_0^\infty(1+x)^{n-1}\cos x\,dx=\Gamma(n)\left\{\cos\left(\frac{n\pi}{2}-1\right)-\frac{1}{\Gamma(n+1)}+\frac{1}{\Gamma(n+3)}-\dots\right\}.$$

(Peterhouse, 1895.)

14. By taking as contour of integration a parabola with its vertex at the origin, derive from the formula

$$\Gamma(a)=-\frac{1}{2i\sin a\pi}\int_\infty^{(0+)}(-z)^{a-1}e^{-z}\,dz$$

the result

$$\Gamma(a)=\frac{1}{2\sin a\pi}\int_0^\infty e^{-x^2}x^{a-1}(1+x^2)^{\frac{1}{2}a}[3\sin\{x+a\operatorname{arc\,cot}(-x)\}$$
$$+\sin\{x+(a-2)\operatorname{arc\,cot}(-x)\}]\,dx,$$

the arc cot denoting an obtuse angle.

(Bourguet, *Acta Math.* I. p. 367.)

15. Shew that, if the real part of a_n is positive and $\sum_{n=1}^{\infty}1/a_n^2$ is convergent, then

$$\prod_{n=1}^{\infty}\left[\frac{\Gamma(a_n)}{\Gamma(z+a_n)}\exp\left\{\sum_{s=1}^{m}\frac{z^s}{s!}\psi^{(s)}(a_n)\right\}\right]$$

is convergent when $m>2$, where $\psi^{(s)}(z)=\dfrac{d^s}{dz^s}\log\Gamma(z)$. (Math. Trip. 1907.)

16. Prove that

$$\frac{d\log\Gamma(z)}{dz}=\int_0^\infty\frac{e^{-a}-e^{-za}}{1-e^{-a}}\,da-\gamma$$

$$=\int_0^\infty\{(1+a)^{-1}-(1+a)^{-z}\}\frac{da}{a}-\gamma$$

$$=\int_0^1\frac{x^{z-1}-1}{x-1}\,dx-\gamma.$$ (Legendre.)

17. Prove that, when $R(z)>0$,

$$\log\Gamma(z)=\int_0^1\left\{\frac{x^z-x}{x-1}-x(z-1)\right\}\frac{dx}{x\log x}.$$ (Binet.)

18. Prove that, for all values of z except negative real values,

$$\log\Gamma(z)=(z-\tfrac{1}{2})\log z-z+\tfrac{1}{2}\log(2\pi)$$
$$+\tfrac{1}{2}\left\{\frac{1}{2\,.\,3}\sum_{r=1}^{\infty}\frac{1}{(z+r)^2}+\frac{2}{3\,.\,4}\sum_{r=1}^{\infty}\frac{1}{(z+r)^3}+\frac{3}{4\,.\,5}\sum_{r=1}^{\infty}\frac{1}{(z+r)^4}+\dots\right\}.$$

19. Prove that, when $R(z)>0$,

$$\frac{d}{dz}\log\Gamma(z)=\log z-\int_0^1\frac{x^{z-1}\,dx}{(1-x)\log x}\{1-x+\log x\}.$$

20. Prove that, when $R(z)>0$,

$$\frac{d^2}{dz^2}\log\Gamma(z)=\int_0^\infty\frac{xe^{-xz}\,dx}{1-e^{-x}}.$$

21. If

$$\int_z^{z+1}\log\Gamma(t)\,dt=u,$$

shew that

$$\frac{du}{dz}=\log z,$$

and deduce from § 12·33 that, for all values of z except negative real values,

$$u=z\log z-z+\tfrac{1}{2}\log(2\pi).$$

(Raabe, *Journal für Math.* XXV.)

22. Prove that, for all values of z except negative real values,

$$\log\Gamma(z)=(z-\tfrac{1}{2})\log z-z+\tfrac{1}{2}\log(2\pi)+\sum_{n=1}^{\infty}\int_0^\infty\frac{dx}{x+z}\,\frac{\sin 2n\pi x}{n\pi}.$$

(Bourguet*.)

23. Prove that

$$B(p,p)\,B(p+\tfrac{1}{2},p+\tfrac{1}{2})=\frac{\pi}{2^{4p-1}p}.$$ (Binet.)

24. Prove that, when $-t<r<t$,

$$B(t+r,t-r)=\frac{1}{4^{t-1}}\int_0^\infty\frac{\cosh(2ru)\,du}{\cosh^{2t}u}.$$

25. Prove that, when $q>1$,

$$B(p,q)+B(p+1,q)+B(p+2,q)+\dots=B(p,q-1).$$

26. Prove that, when $p-a>0$,

$$\frac{B(p-a,q)}{B(p,q)}=1+\frac{aq}{p+q}+\frac{a(a+1)\,q\,(q+1)}{1\,.\,2\,.\,(p+q)(p+q+1)}+\dots.$$

27. Prove that

$$B(p,q)\,B(p+q,r)=B(q,r)\,B(q+r,p).$$ (Euler.)

28. Shew that

$$\int_0^1 x^{a-1}(1-x)^{b-1}\frac{dx}{(x+p)^{a+b}}=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,\frac{1}{(1+p)^a p^b},$$

if $a>0$, $b>0$, $p>0$. (Trinity, 1908.)

* This result is attributed to Bourguet by Stieltjes, *Journal de Math.* (4), v. p. 432.

29. Shew that, if $m>0$, $n>0$, then

$$\int_{-1}^{1} \frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\,dx = 2^{m+n-2}\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)};$$

and deduce that, when a is real and not an integer multiple of $\frac{1}{2}\pi$,

$$\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)^{\cos 2a} d\theta = \frac{\pi}{2\sin(\pi\cos^2 a)}.$$

(St John's, 1904.)

30. Shew that, if $a>0$, $\beta>0$,

$$\int_0^1 \frac{t^{a-1}}{1+t}\,dt = \tfrac{1}{2}\psi\left(\tfrac{1}{2}+\tfrac{1}{2}a\right) - \tfrac{1}{2}\psi\left(\tfrac{1}{2}a\right),$$

and

$$\int_0^1 \frac{t^{a-1}-t^{\beta-1}}{(1+t)\log t}\,dt = \log\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}\beta\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta\right)}.$$

(Kummer.)

31. Shew that, if $a>0$, $a+b>0$,

$$\int_0^1 \frac{x^{a-1}(1-x^b)}{1-x}\,dx = \lim_{\delta\to 0}\left\{\frac{\Gamma(a)\,\Gamma(\delta)}{\Gamma(a+\delta)} - \frac{\Gamma(a+b)\,\Gamma(\delta)}{\Gamma(a+b+\delta)}\right\} = \psi(a+b)-\psi(a).$$

Deduce that, if in addition $a+c>0$, $a+b+c>0$,

$$\int_0^1 \frac{x^{a-1}(1-x^b)(1-x^c)}{(1-x)(-\log x)}\,dx = \log\frac{\Gamma(a)\,\Gamma(a+b+c)}{\Gamma(a+b)\,\Gamma(a+c)}.$$

32. Shew that, if a, b, c be such that the integral converges,

$$\int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\log x)}\,dx = \log\frac{\Gamma(b+c+1)\,\Gamma(c+a+1)\,\Gamma(a+b+1)}{\Gamma(a+1)\,\Gamma(b+1)\,\Gamma(c+1)\,\Gamma(a+b+c+1)}.$$

33. By the substitution $\cos\theta = 1-2\tan\frac{1}{2}\phi$, shew that

$$\int_0^\pi \frac{d\theta}{(3-\cos\theta)^{\frac{1}{2}}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{\pi}}.$$

(St John's, 1896.)

34. Evaluate in terms of Gamma-functions the integral $\int_0^\infty \frac{\sin^p x}{x}\,dx$, when p is a fraction greater than unity whose numerator and denominator are both odd integers.

[Shew that the integral is $\frac{1}{2}\int_0^\pi \sin^p x\left\{\frac{1}{x}+\sum_{n=1}^{\infty}(-)^n\left(\frac{1}{x+n\pi}+\frac{1}{x-n\pi}\right)\right\}dx$.]

(Clare, 1898.)

35. Shew that

$$\int_0^{\frac{1}{2}\pi} \left(1-\tfrac{1}{2}\sin^2 x\right)^{n-\frac{1}{2}}dx = \frac{n\,!}{2^{n+2}\pi^{\frac{1}{2}}}\sum_{r=0}^{n}\frac{2^{3r}}{2r\,!\,(n-r)\,!}\left\{\Gamma\left(\frac{2r+1}{4}\right)\right\}^2.$$

36. Prove that

$$\log B(p,q) = \log\left(\frac{p+q}{pq}\right) + \int_0^1 \frac{(1-v^p)(1-v^q)}{(1-v)\log v}\,dv.$$

(Euler.)

37. Prove that, if $p>0$, $p+s>0$, then

$$B(p,p+s) = \frac{B(p,p)}{2^s}\left\{1+\frac{s(s-1)}{2(2p+1)}+\frac{s(s-1)(s-2)(s-3)}{2\,.\,4\,.\,(2p+1)(2p+3)}+\ldots\right\}.$$

(Binet.)

38. The curve $r^m = 2^{m-1}a^m\cos m\theta$ is composed of m equal closed loops. Shew that the length of the arc of half of one of the loops is

$$m^{-1}a\int_0^{\frac{1}{2}\pi}\left(\tfrac{1}{2}\cos x\right)^{\frac{1}{m}-1}dx,$$

and hence that the total perimeter of the curve is

$$a\left\{\Gamma\left(\frac{1}{2m}\right)\right\}^2\Big/\Gamma\left(\frac{1}{m}\right).$$

39. Draw the straight line joining the points $\pm i$, and the semicircle of $|z|=1$ which lies on the right of this line. Let C be the contour formed by indenting this figure at $-i, 0, i$. By considering $\int_C z^{p-q-1}(z+z^{-1})^{p+q-2}\,dz$, shew that, if $p+q>1$, $q<1$,

$$\int_0^{\frac{1}{2}\pi} \cos^{p+q-2}\theta \cos(p-q)\theta\, d\theta = \frac{\pi}{(p+q-1)\, 2^{p+q-1} B(p, q)}.$$

Prove that the result is true for all values of p and q such that $p+q>1$.

(Cauchy.)

40. If s is positive (not necessarily integral), and $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$, shew that

$$\cos^s x = \frac{1}{2^{s-1}} \frac{\Gamma(s+1)}{\{\Gamma(\frac{1}{2}s+1)\}^2} \left\{\frac{1}{2} + \frac{s}{s+2}\cos 2x + \frac{s(s-2)}{(s+2)(s+4)}\cos 4x + \dots\right\},$$

and draw graphs of the series and of the function $\cos^s x$.

41. Obtain the expansion

$$\cos^s x = \frac{a}{2^{s-1}}\Gamma(s+1)\left[\frac{\cos ax}{\Gamma(\frac{1}{2}s+\frac{1}{2}a+1)\Gamma(\frac{1}{2}s-\frac{1}{2}a+1)} + \frac{\cos 3ax}{\Gamma(\frac{1}{2}s+\frac{3}{2}a+1)\Gamma(\frac{1}{2}s-\frac{3}{2}a+1)} + \dots\right],$$

and find the values of x for which it is applicable. (Cauchy.)

42. Prove that, if $p>\frac{1}{2}$,

$$\Gamma(2p) = \frac{2^{2p-1}}{\sqrt{\pi}}\{\Gamma(p)\}^2\left[\frac{2p^2}{2p+1}\left\{1+\frac{1^2}{2(2p+3)} + \frac{1^2.3^2}{2.4.(2p+3)(2p+5)} + \dots\right\}\right]^{\frac{1}{2}}.$$

(Binet.)

43. Shew that, if $x<0$, $x+z>0$, then

$$\frac{\Gamma(-x)}{\Gamma(z)}\left\{\frac{-x}{z} + \frac{1}{2}\frac{(-x)(1-x)}{z(1+z)} + \frac{1}{3}\frac{(-x)(1-x)(2-x)}{z(1+z)(2+z)} + \dots\right\}$$
$$= \frac{1}{\Gamma(x+z)}\int_0^1 t^{-x-1}\{-\log(1-t)\}(1-t)^{x+z-1}\,dt,$$

and deduce that, when $x+z>0$,

$$\frac{d}{dz}\log\frac{\Gamma(z+x)}{\Gamma(z)} = \frac{x}{z} - \frac{1}{2}\frac{x(x-1)}{z(z+1)} + \frac{1}{3}\frac{x(x-1)(x-2)}{z(z+1)(z+2)} - \dots.$$

44. Using the result of example 43, prove that

$$\log\Gamma(z+a) = \log\Gamma(z) + a\log z - \frac{a-a^2}{2z}$$
$$- \sum_{n=1}^{\infty} \frac{a\int_0^1 t(1-t)(2-t)\dots(n-t)\,dt - \int_0^a t(1-t)(2-t)\dots(n-t)\,dt}{(n+1)z(z+1)(z+2)\dots(z+n)},$$

investigating the region of convergence of the series.

(Binet, *Journal de l'École polytechnique*, XVI. (1839), p. 256.)

45. Prove that, if $p>0$, $q>0$, then

$$B(p, q) = \frac{p^{p-\frac{1}{2}} q^{q-\frac{1}{2}}}{(p+q)^{p+q-\frac{1}{2}}}(2\pi)^{\frac{1}{2}} e^{M(p, q)},$$

where

$$M(p, q) = 2\rho \int_0^\infty \frac{dt}{e^{2\pi t\rho} - 1} \arc\tan \left\{ \frac{(t^3 + t)\rho^3}{pq(p+q)} \right\},$$

and

$$\rho^2 = p^2 + q^2 + pq.$$

46. If $$U = 2^{\frac{1}{2}x}/\Gamma(1 - \tfrac{1}{2}x), \quad V = 2^{\frac{1}{2}x}/\Gamma(\tfrac{1}{2} - \tfrac{1}{2}x),$$

and if the function $F(x)$ be defined by the equation

$$F(x) = \pi^{\frac{1}{2}} \left(V \frac{dU}{dx} - U \frac{dV}{dx} \right),$$

shew (1) that $F(x)$ satisfies the equation

$$F(x+1) = xF(x) + \frac{1}{\Gamma(1-x)},$$

(2) that, for all positive integral values of x,

$$F(x) = \Gamma(x),$$

(3) that $F(x)$ is analytic for all finite values of x,

(4) that $$F(x) = \frac{1}{\Gamma(1-x)} \frac{d}{dx} \log \frac{\Gamma\left(\frac{1-x}{2}\right)}{\Gamma\left(1 - \frac{x}{2}\right)}.$$

47. Expand

$$\{\Gamma(a)\}^{-1}$$

as a series of ascending powers of a.

(Various evaluations of the coefficients in this expansion have been given by Bourguet, *Bull. des Sci. Math.* v. (1881), p. 43; Bourguet, *Acta Math.* II. (1883), p. 261; Schlömilch, *Zeitschrift für Math. und Phys.* xxv. (1880), pp. 35, 351.)

48. Prove that the G-function, defined by the equation

$$G(z+1) = (2\pi)^{\frac{1}{2}z} e^{-\frac{1}{2}z(z+1) - \frac{1}{2}\gamma z^2} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^n e^{-z + z^2/(2n)} \right\},$$

is an integral function which satisfies the relations

$$G(z+1) = \Gamma(z)\, G(z), \quad G(1) = 1,$$

$$(n!)^n / G(n+1) = 1^1 . 2^2 . 3^3 \ldots n^n. \qquad \text{(Alexeiewsky.)}$$

(The most important properties of the G-function are discussed in Barnes' memoir, *Quarterly Journal*, XXXI.)

49. Shew that

$$\frac{G'(z+1)}{G(z+1)} = \tfrac{1}{2}\log(2\pi) + \tfrac{1}{2} - z + z\frac{\Gamma'(z)}{\Gamma(z)},$$

and deduce that

$$\log \frac{G(1-z)}{G(1+z)} = \int_0^z \pi z \cot \pi z \, dz - z \log(2\pi).$$

50. Shew that

$$\int_0^z \log \Gamma(t+1)\, dt = \tfrac{1}{2}z \log(2\pi) - \tfrac{1}{2}z(z+1) + z \log \Gamma(z+1) - \log G(z+1).$$