I. Class Business

A. Take up HW#2

B. Assign HW#3 Due 9/11/01

#’s 2.2, 2.3, 2.4

Read §§ 1.8, 1.9, 1.10, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6

II. Lecture Outline

A. Boundary Conditions & Uniqueness of Solution to Poisson Equation

B. Image Charges
2. Boundary conditions:

\[ \Sigma_2 = \partial V_3 \]

\[ \partial V_3 \text{ could = "00" or could be in } \Sigma_2 \]

2-Types of Boundary Conditions

1. Dirichlet: give \( \phi \) in \( V_3 \) + \( \phi \) on \( \partial V_3 \)

2. Neumann: give \( \phi \) in \( V_3 \) and \( \hat{n} \cdot \nabla \phi \) on \( \partial V_3 \)

(\( \hat{n} = \text{normal to } \partial V_3 \))
We want to show that given either Dirichlet or Neumann Boundary Conditions, the solution to electrostatics is unique.

\[ \nabla^2 \varphi = -\frac{4\pi}{\varepsilon} \frac{\rho}{r} \]

See § 1.9. Assume you have 2 solutions

\[ U = \varphi_2 - \varphi_1 \]

\[ \int_{V_3} \nabla \cdot (\nabla \varphi) \, d^3r = \int_{\partial V_3} \nabla \varphi \cdot d\vec{a} \]

\[ \int_{V_3} \left[ \nabla^2 \varphi + \frac{\rho}{\varepsilon} \right] \, d^3r = \int_{\partial V_3} \nabla \varphi \cdot d\vec{a} \]

\[ \int_{V_3} |\nabla \varphi|^2 \, d^3r = 0 \]

\[ \Rightarrow \nabla \cdot \nabla \varphi = 0 \Rightarrow U = \text{const. in } V_3 \]
For the above argument to work $\nabla \Phi$ must be continuous, i.e. $\phi_1 + \phi_2$ must have exactly the same discontinuities.

$$\phi$$

\[ \Delta \phi = \text{step} \quad \nabla^2 \phi = \delta \text{ for source} \]

Follows from:

$$\frac{d}{dx} \Theta(x-x_0) = \delta(x-x_0)$$

$$\int_{-\infty}^{\infty} S(x) \frac{d}{dx} \Theta(x-x_0) dx = s(x) \Theta(x-x_0) \bigg|_{-\infty}^{\infty} = f(x_0)$$

Test function

$$\frac{d}{dx} \Theta(x-x_0) \text{ has the properties of } \delta(x-x_0)$$

\[ \int_{-\infty}^{\infty} \frac{df(x)}{dx} dx = f(x_0) - f(-\infty) \]
B. Image Charges: \( \S 20.1 \)

\[ (1) \]

\[ \begin{aligned}
\mathbf{V}_2 & \rightarrow \mathbf{V}_3 \\
(0,0,2\rho) & \quad \partial \mathbf{V}_3
\end{aligned} \]

"\( \infty \)" grounded conductor

\[ \nabla^2 \Phi = -\frac{4\pi q}{\varepsilon_0} \delta(r - 2\rho \hat{r}) \]

\[ \Phi = 0 \quad \text{for} \quad \mathbf{r} = (x, y, 0) \]

\[ |\mathbf{r}| \rightarrow \infty \quad \text{with} \quad \varepsilon > 0 \]

Dischlet Boundary conditions!

Unique answer:

\[ \Phi = \frac{q}{4\pi\varepsilon_0 \sqrt{x^2 + y^2 + (z - 2\rho)^2}} - \frac{q}{4\pi\varepsilon_n \sqrt{x^2 + y^2 + (z + 2\rho)^2}} \]
This \( \Phi \) satisfies \( \Psi(x_0 y_0) = 0 \) \( \Phi = 0 \) at \( r \to \infty \) \( (z > 0) \)

and \( \nabla^2 \Phi = -\frac{4\pi}{\varepsilon_0} g \delta(r^2 - z^2 \bar{r}^2) \) in \( V_2 \)

(2) \( \Phi \) : Gounded Sphere!

\[ \Phi = \frac{q}{4\pi\varepsilon_0 |x - \bar{x}|} + \hat{\Phi} \]

\[ \nabla^2 \Phi = -\frac{4\pi}{\varepsilon_0} g \delta(x - \bar{x}) \]

\( \Phi \) is absent inside sphere

\( \Phi \) is absent from symmetry
\[ \bar{x} = x \bar{m}, \quad \bar{y} = y \bar{m}', \quad \bar{y}' = y' \bar{m}' \quad (2.3) \]

\[ \therefore \bar{m} = \frac{9}{4} \frac{1}{\bar{m}} + \frac{9}{4} \frac{1}{\bar{m}'} \]

\[ \text{if}\ (r = 0) = 0 \Rightarrow \quad \frac{9}{1} \frac{1}{a^2 - y^2} = 0 \]

\[ 0 = \frac{9}{a^2 + y^2 - 2ay \bar{m}} + \frac{9}{a^2 + y^2 - 2ay \bar{m}'} \]

\[ \bar{y} \text{ has opposite sign of } y \]

\[ 9^2 (a^2 + y^2 - 2ay \bar{m}) = (9')^2 (a^2 + y^2 - 2ay \bar{m}') \quad \forall \bar{m} \]
From (2) \[ q^2(a^2 + y'^2) = (y')^2(a^2 + y^2) \]
and \[ q^2(-2ay') = y'^2(-2ay) \]
2 unknowns \( q', y' \) and 2 equations!

Ratio of above eqns \( \Rightarrow \)

\[ \frac{y'}{a^2 + y'^2} = \frac{y}{a^2 + y^2} \Rightarrow y y' - (a^2 y') y' + a^2 y = 0 \]

quadratic eqn

\[ y' = \frac{a^2 + y^2 \pm \sqrt{(a^2 y')^2 - 4 a^2 y^2}}{2 y} \]

\[ = \frac{(a^2 + y^2) \pm (a^2 - a^2)}{2 y} = y \sqrt{\frac{a^2}{y}} \]

second eqn. \( \Rightarrow \) \[ \frac{q'}{q} = -\sqrt{\frac{y'}{y}} = -\frac{a}{y} \]
\[ \lim_{a \to \infty} = ? \quad y - a = d \quad \text{find} \]

\[ = ) \quad a - y' = a - \frac{a^2}{y} = a \left( \frac{y-a}{y} \right) \]

\[ = a \left( \frac{d}{a+d} \right) \to d \]

\[ g' = -\frac{a}{y} \quad \theta = -\frac{a}{d+a} \quad g \to -f \]

- Same as minor - i.e. grounded flat plane.