

Wednesday January 19, 2005  
quiz #1

**Problem 0:** Write down Maxwell's equations in differential form.  
solution:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_o \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_o \vec{J} + \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}\end{aligned}$$

**Problem 1:** The function  $f$  is given by

$$f(x, y, z) = \tanh(x^2 - \sin(yz)).$$

Evaluate the integral

$$\int_{\vec{r}_A}^{\vec{r}_B} (\vec{\nabla} f) \cdot d\vec{\ell}$$

over the arc described by the parameterization

$$\vec{r}' = a \cos \theta \hat{i} + b \sin \theta \hat{j}, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

**solution:**

By the fundamental theorem for gradients, we have

$$\begin{aligned}\int_{\vec{r}_A}^{\vec{r}_B} (\vec{\nabla} f) \cdot d\vec{\ell} &= f(\vec{r}_B) - f(\vec{r}_A) \\ \vec{r}_A &= a\hat{i}, \quad f(\vec{r}_A) = \tanh(a^2) \\ \vec{r}_B &= a\hat{j}, \quad f(\vec{r}_B) = \tanh(0) = 0 \\ \int_{\vec{r}_A}^{\vec{r}_B} (\vec{\nabla} f) \cdot d\vec{\ell} &= -\tanh(a^2)\end{aligned}$$

**Problem 2:** The vector field  $\vec{v}$  is given by

$$\vec{v} = (x^2 + y^2 + z^2 - a^2)(\cos \theta \hat{x} + \sin \theta \hat{y}) + (x^2 + y^2 + z^2)(\cos \phi \sin \theta \hat{x} + \sin \phi \sin \theta \hat{y} + \cos \theta \hat{z})$$

evaluate the integral

$$\iiint_V (\vec{\nabla} \cdot \vec{v}) d\tau$$

where  $V$  is the volume of the interior of a sphere of radius  $a$  centered at the origin.

**solution:**

By Gauss' law we have

$$\iiint_V (\vec{\nabla} \cdot \vec{v}) d\tau = \oiint \vec{v} \cdot d\vec{a}$$

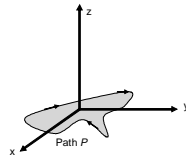
but on the surface  $r^2 = a^2$  so  $\vec{v}$  becomes

$$\begin{aligned} \vec{v} &= a^2(\cos \phi \sin \theta \hat{x} + \sin \phi \sin \theta \hat{y} + \cos \theta \hat{z}) = a^2 \hat{r} \\ d\vec{a} &= \hat{r} a^2 d \cos \theta d\phi \end{aligned}$$

so we have

$$\iiint_V (\vec{\nabla} \cdot \vec{v}) d\tau = \oiint \vec{v} \cdot d\vec{a} = 4\pi a^3$$

**Problem 3:** The path  $P$  of the figure below lies completely in the  $x - y$  plane and encloses an area  $A$ .



Consider the vector field

$$\vec{v} = y\hat{x} + z\hat{y} + x\hat{z}.$$

Evaluate

$$\oint_P \vec{v} \cdot d\vec{\ell}$$

**solution:**

By Stokes theorem we have

$$\oint_P \vec{v} \cdot d\vec{\ell} = \iint \vec{\nabla} \times \vec{v} \cdot d\vec{a}$$

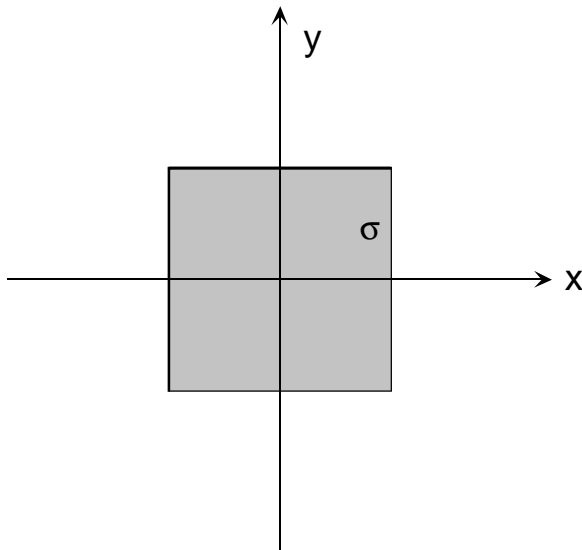
But

$$\begin{aligned} \vec{\nabla} \times \vec{v} &= -\hat{j} + \hat{k} \\ d\vec{a} &= -\hat{k} da \end{aligned}$$

so

$$\oint_P \vec{v} \cdot d\vec{\ell} = - \iint da = -A$$

**Problem 4.** Find an exact expression for the field everywhere on the  $x$ -axis if a square of dimension  $a$  and uniform charge per unit area  $\sigma$  sits centered on the  $x - y$  plane as shown:



Hint: Find the electric field directly without first computing  $V(\vec{r})$  and use one of the given integrals. Do not expect this to simplify into something pretty. Sometimes the answer is just ugly.

**solution:**

Because  $\vec{\nabla} \times \vec{E} = \vec{0}$  and  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ , we can write

$$\begin{aligned} \vec{E} &= \frac{1}{4\pi\epsilon_0} \iint \sigma \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} da' \\ \vec{r}' &= (x', y'), \vec{r} = (x, 0), da' = dx' dy' \end{aligned}$$

By symmetry we have  $\vec{E} = E_x \hat{i}$  with

$$E_x = \frac{\sigma}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \frac{(x-x')}{((x-x')^2 + y'^2)^{3/2}} dx' dy'$$

The integral over  $x'$  is simple whereas the integral over  $y'$  is done with the help of a standard integral

$$E_x = \frac{\sigma}{4\pi\epsilon_0} \left[ \ln \left[ \frac{\sqrt{2a^2 - 4ax + 4x^2} + a}{\sqrt{2a^2 - 4ax + 4x^2} - a} \right] - \ln \left[ \frac{\sqrt{2a^2 + 4ax + 4x^2} + a}{\sqrt{2a^2 + 4ax + 4x^2} - a} \right] \right]$$

Although certainly not required, when it is a good idea to do a Taylor series of this solution to make sure it makes sense. Suppose you type this solution into Mathematica. You know that  $1/x$  is what is getting small, so the command

$$\left( \text{Series}[E_x/.{x->\frac{1}{\Delta}}, \{\Delta, 0, 5\}] \right) /. \Delta -> \frac{1}{x}$$

will do the trick. Inside the series command,  $E_x$  is replaced by a function in which  $x$  is replaced by  $1/\Delta$ . The Series command then creates a Taylor series in  $\Delta$ . Finally, the last substitution replaces  $\Delta$  with  $1/x$ . The resulting output is

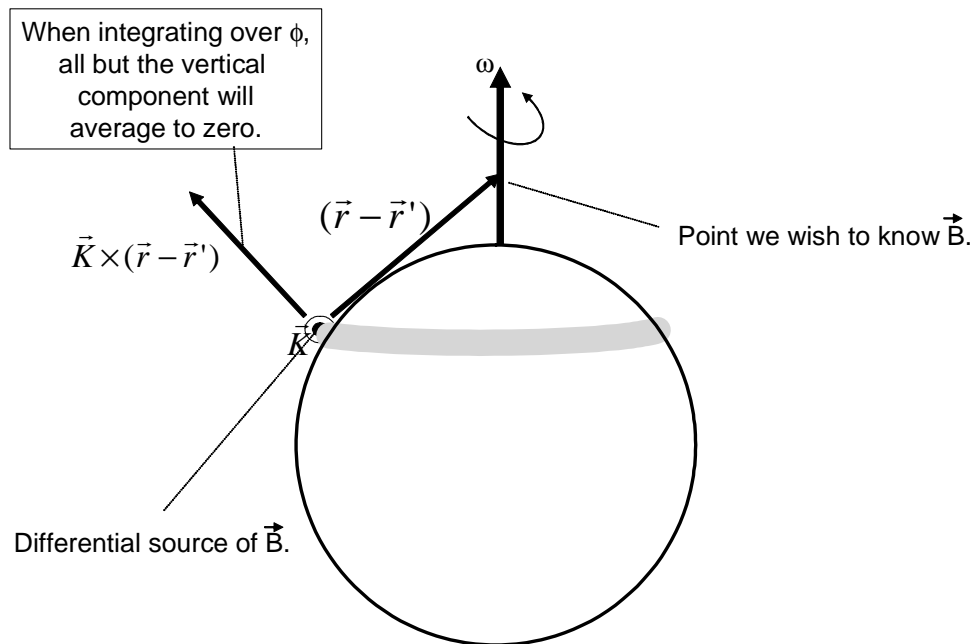
$$E_x = \frac{a^2\sigma}{4\pi\epsilon_0 x^2} + \frac{a^4\sigma}{24\pi\epsilon_0 x^4} + O\left(\frac{1}{x^6}\right)$$

We can see immediately that our solution has the right limiting form, that is as  $x$  becomes much bigger than  $a$ , our value for  $E_x$  approaches  $\frac{Q}{4\pi\epsilon_0 r^2}$ .

**Problem 5** (a) (a few points) Use ampere's law to find the field a distance  $s$  from a wire carrying a current  $I$ .

$$\vec{B} = \frac{\mu_0 I}{2\pi s}$$

(b) (lots of points) A sphere of radius  $a$  carries an surface charge density  $\sigma = \sigma_0 / \sin^2 \theta'$  in the range of spherical polar angles  $\frac{\pi}{4} < \theta' < \frac{\pi}{2}$  and  $\sigma = 0$  for all other angles. The sphere spins with an angular velocity  $\omega$  about a  $z$ -axis as shown.



Use the Biot-Savart law to find  $\vec{B}$  on the  $z$  axis above the spinning sphere. (Hint:  $\vec{K}' = \sigma \vec{v}'$ . Express the Cartesian components of  $\vec{v}'$  in terms of  $\omega$ ,  $a \sin \theta'$ , and  $\phi'$ .)

Solution: First we find  $\vec{K}'$

$$\begin{aligned} \vec{v}' &= \vec{\omega} \times \vec{r}', \quad \vec{\omega} = (0, 0, \omega) \\ \vec{r}' &= a(\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta') \\ \vec{v}' &= a\omega \sin \theta'(-\sin \phi', \cos \phi', 0) \\ \vec{K}' &= \sigma \vec{v}' = \sigma a \omega \sin \theta'(-\sin \phi', \cos \phi', 0) \end{aligned}$$

Now we have

$$\begin{aligned}
\vec{B} &= \frac{\mu_o}{4\pi} \iint \frac{\vec{K}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} da' \\
\vec{r} - \vec{r}' &= (-a \sin \theta' \cos \phi', -a \sin \theta' \sin \phi', z - a \cos \theta') \\
\vec{K}' \times (\vec{r} - \vec{r}') &= [\hat{\theta} + \sigma \omega a^2 \sin^2 \theta' \hat{z} = \sigma_o \omega a^2 \hat{z} \\
\vec{B} &= \frac{\mu_o \sigma_o \omega a^4}{2} \hat{z} \int_0^{1/\sqrt{2}} \frac{1}{(a^2 + z^2 - 2az \cos \theta')^{3/2}} d \cos \theta' \\
\int_0^{2^{-1/2}} \frac{1}{(a^2 + z^2 - 2azx)^{3/2}} dx &= \frac{1}{az(a^2 + z^2 - 2azx)^{1/2}} \Big|_0^{1/\sqrt{2}} \\
&= \frac{1}{az^2} \left[ \frac{z}{(z^2 + a^2 - 2^{1/2}az)^{1/2}} - \frac{z}{(z^2 + a^2)^{1/2}} \right] \\
\vec{B} &= \frac{\mu_o \sigma_o \omega a^3}{2z^2} \left[ \frac{z}{(z^2 + a^2 - 2^{1/2}az)^{1/2}} - \frac{z}{(z^2 + a^2)^{1/2}} \right] \hat{z}
\end{aligned}$$

(c) (a few points) Substitute  $I = \sigma_o \omega L^2$  into (a) to make sure the units of your answer to (a) agrees with the units of your solution to (b).  
solution:

This is just one quick check. We know  $B = \frac{\mu_o I}{2\pi s}$  which has units  $\frac{\mu_o \sigma_o \omega L^2}{s}$  which has the same units as  $\mu_o \sigma_o \omega$  times a length. This agrees with the units of our solution above.

(d) (a few points) For large  $z$ , how fast does your expression for the  $\vec{B}$  field given in (b) fall off with  $z$ ?

This is another check. For large  $z$ ,

$$\begin{aligned}
\left[ \frac{z}{(z-a)} - \frac{z}{(z^2 + a^2 - 2^{1/2}az)^{1/2}} \right] &= \left[ \frac{1}{(1 + (\frac{a}{z})^2 - 2^{1/2}\frac{a}{z})^{1/2}} - \frac{1}{(1 + (\frac{a}{z})^2)^{1/2}} \right] \\
&= \frac{a}{2^{1/2}z}
\end{aligned}$$

so we have

$$\vec{B} \approx \frac{\mu_o \sigma_o \omega a^4}{2^{3/2} z^3} \hat{z} \text{ at } \vec{r} = (0, 0, z) \text{ with } z \gg a.$$

Note that  $\vec{B}$  must fall faster than  $z^2$  because there is no magnetic monopole.

**Problem 6.** A sphere of radius  $R$  carries a free charge density  $\rho = \rho_o (r/R)^{1/2}$ . The sphere is made of a dielectric material of electric susceptibility  $\chi_e$ . Find

(a) The displacement vector  $\vec{D}$  inside and outside the sphere

$\vec{\nabla} \cdot \vec{D} = \rho_f$  so we can use Gauss' law, which for a sphere gives

$$\begin{aligned}
 4\pi r^2 D &= 4\pi \int_0^r \rho r'^2 dr' \\
 &= \frac{4\pi\rho_o}{R^{1/2}} \int_0^{\min(r,R)} r'^{5/2} dr' \\
 &= \frac{8\pi\rho_o}{7R^{1/2}} (\min(r,R))^{7/2} \\
 D &= \frac{2\rho_o}{7R^{1/2}} \frac{(\min(r,R))^{7/2}}{r^2} \\
 \vec{D} &= \begin{cases} \frac{2\rho_o R}{7} \left(\frac{r}{R}\right)^{3/2} \hat{r} & r < R \\ \frac{2\rho_o R}{7} \left(\frac{R}{r}\right)^2 \hat{r} & r > R \end{cases}
 \end{aligned}$$

(b) The polarization  $\vec{P}$  inside the sphere.

For a linear dielectric, we have

$$\vec{P} = \frac{\chi_e}{1 + \chi_e} \vec{D} = \begin{cases} \frac{2\rho_o R \chi_e}{7(1 + \chi_e)} \left(\frac{r}{R}\right)^{3/2} \hat{r} & r < R \\ \vec{0} & r > R \end{cases}$$

(c) The bound charge density and bound surface charge density. (Hint: Make sure  $\rho_b$  has the units of  $\rho_o$  and  $\sigma_b$  has the units of  $\rho_o R$ .)

$$\begin{aligned}
 \rho_b &= -\vec{\nabla} \cdot \vec{P} \\
 &= -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 P_r \\
 &= \frac{-\rho_o \chi_e}{(1 + \chi_e)} \left(\frac{r}{R}\right)^{1/2} \\
 \sigma_b &= \hat{n} \cdot \vec{P} = \frac{2\rho_o R \chi_e}{7(1 + \chi_e)}
 \end{aligned}$$

**Problem 7.** An infinitely long cylinder of radius  $a$  lies on the  $z$  axis of a coordinate system and carries a uniform magnetization  $\vec{M} = M_o \hat{i}$ . (NOTE: The axis of polarization is at right angles to the axis of the cylinder! You will need some of the integrals given on page 1.)

(a) Find the bound current  $\vec{J}_b$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \vec{0}$$

(b) Find the bound surface current  $\vec{K}_b$

$$\vec{K}_b = \vec{M} \times \hat{n}$$

But

$$\hat{n} = \cos \phi' \hat{i} + \sin \phi' \hat{j}$$

)So

$$\vec{K}_b = M_o \sin \phi' \hat{k}$$

- (c) Find the  $\vec{B}$  field at a point  $x\hat{i}$  directly from your answers to parts a and b. Assume  $x > a$ .

$$\vec{B} = \frac{\mu_o}{4\pi} \iint \frac{\vec{K}_b \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} da'$$

$$\begin{aligned} \vec{r} - \vec{r}' &= (x - a \cos \phi')\hat{i} - a \sin \phi' \hat{j} - z' \hat{k} \\ \vec{K}_b \times (\vec{r} - \vec{r}') &= M_o(x - a \cos \phi')\hat{j} + M_o a \sin \phi' \hat{i} \\ \vec{B} &= \frac{\mu_o M_o}{4\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{(x - a \cos \phi') \sin \phi' \hat{j} + a \sin^2 \phi' \hat{i}}{(x^2 + a^2 - 2ax \cos \phi' + z'^2)^{3/2}} dz' a d\phi' \\ &= \frac{\mu_o M_o a}{2\pi} \int_0^{2\pi} \frac{(x - a \cos \phi') \sin \phi' \hat{j} + a \sin^2 \phi' \hat{i}}{(x^2 + a^2 - 2ax \cos \phi')} d\phi' \quad (\text{integral easily solved with mathematic}) \\ &= \frac{\mu_o M_o a^2}{2x^2} \hat{i} \quad (\text{note, just to really screw you up, the integral in the table was wrong.}) \end{aligned}$$

note: to solve the integral with mathematica, it is important to let it know as much as possible about the range of the variables. The right line is

$$\text{Simplify}\left[\frac{\mu M a}{2\pi} \text{Integrate}\left[\frac{a \sin^2 \phi}{x^2 + a^2 - 2ax \cos \phi}, \{\phi, 0, 2\pi\}, \text{Assumptions} \rightarrow \{a > 0, x > a\}\right]\right]$$

I encourage you to type this in and try it for kicks. Then try again without the Assumptions statement.

- (d) Find  $\vec{H}$  at a point  $x\hat{i}$  with  $x > a$  directly from  $\rho_m = -\vec{\nabla} \cdot \vec{M}$ ,  $\sigma_m = \vec{M} \cdot \hat{n}$  and show  $\vec{B} = \mu_o \vec{H} + \vec{M}$  agrees with your answer to part c.

Here we have

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} = \vec{0}$$

and

$$\vec{\nabla} \cdot \vec{H} = \frac{1}{\mu_o} \vec{\nabla} \cdot \vec{B} - \vec{\nabla} \cdot \vec{M} = -\vec{\nabla} \cdot \vec{M}$$

but, if we take  $\rho_m = -\vec{\nabla} \cdot \vec{M}$ ,  $\sigma_m = \vec{M} \cdot \hat{n}$ , These are exactly the conditions on the electric field given a charge distribution. So, letting

$$\begin{aligned} \rho_m &= -\vec{\nabla} \cdot \vec{M} = 0 \\ \sigma_m &= \vec{M} \cdot \hat{n} = M_o \cos \phi' \end{aligned}$$

we have

$$\begin{aligned}
\vec{H} &= \frac{1}{4\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} M_o \cos \phi' \frac{((x - a \cos \phi')\hat{i} - a \sin \phi' \hat{j} - z' \hat{k})}{(x^2 - 2ax \cos \phi' + a^2 + z'^2)^{3/2}} adz' d\phi' \\
&= \frac{M_o a}{2\pi} \int_0^{2\pi} \frac{(x - a \cos \phi') \cos \phi' \hat{i}}{(x^2 - 2ax \cos \phi' + a^2)} d\phi' \\
&= \frac{M_o a^2}{2x^2} \hat{i} \quad (\text{I used mathematica to do this integral}) \\
\vec{H} &= \frac{1}{\mu_o} \vec{B} + \vec{M} \quad (\vec{M} = \vec{0} \text{ outside the cylinder.}) \\
\vec{B} &= \mu_o \vec{H} = \frac{\mu_o a^2}{2x^2} M_o \hat{i}
\end{aligned}$$