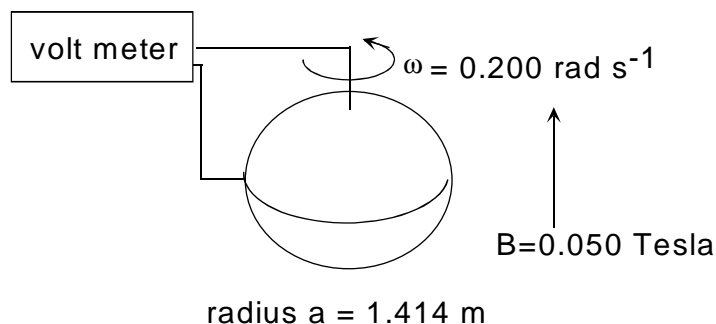


Problem 1: Strange power plant.



A perfectly conducting spherical shell of radius a rotates in along an axis parallel to a uniform magnetic field \vec{B} at an angular frequency ω . A voltmeter is hooked up to the north pole and equator (using conductive brushes.) Find an expression for the voltage read in terms of B_o , ω and a and find the voltage for the values indicated in the figure.

In this problem we have

$$\xi = \int \vec{f} \cdot d\vec{\ell}$$

with \vec{f} the sum of all non-resistive forces per charge. Here this will be given by $\vec{v} \times \vec{B}$ with $\vec{v} = \vec{r} \times \vec{\omega}$. Thus

$$\begin{aligned} \vec{f} &= -\vec{B} \times (\vec{r} \times \vec{\omega}) \\ &= -(\vec{B} \cdot \vec{r})\vec{\omega} + (\vec{B} \cdot \vec{\omega})\vec{r} \end{aligned}$$

Now we let $\vec{r}(\theta)$ be any path between the two terminals of the sphere. We let

$$\vec{r} = a(\sin \theta \hat{x} + \cos \theta \hat{z})$$

so that

$$\begin{aligned} \vec{f} &= -B_o a \omega \cos \theta \hat{z} + B_o a \omega (\sin \theta \hat{x} + \cos \theta \hat{z}) \\ &= B_o a \omega \sin \theta \hat{x} \\ d\vec{\ell} &= a(\cos \theta, 0, -\sin \theta) d\theta \end{aligned}$$

leading to

$$\begin{aligned} \xi &= B_o a^2 \omega \int_0^{\pi/2} \sin \theta \cos \theta d\theta \\ &= \frac{B_o a^2 \omega}{2} \int_0^{\pi/2} \sin 2\theta d\theta \\ &= \frac{B_o a^2 \omega}{4} = \frac{(0.05 \text{ Tesla})(2 \text{ m})^2 (.2 \text{ s}^{-1})}{4} = 5 \text{ mV} \end{aligned}$$

Problem 2: Not about to change space travel

We will learn in class that far away from a time-dependent neutral charge distribution, the electric and magnetic fields are often well approximated by

$$\vec{E} = \frac{\mu_o}{4\pi r} [\hat{r} \times (\hat{r} \times \frac{d^2 \vec{p}}{dt^2})] \quad (1)$$

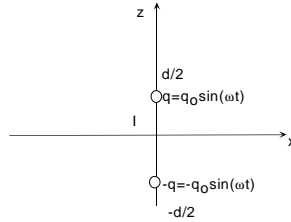
$$= \frac{\mu_o}{4\pi r} \left[\hat{r} (\hat{r} \cdot \frac{d^2 \vec{p}}{dt^2}) - \frac{d^2 \vec{p}}{dt^2} \right] \quad (2)$$

$$\vec{B} = \frac{-\mu_o}{4\pi r c} [\hat{r} \times \frac{d^2 \vec{p}}{dt^2}] \quad (3)$$

where \vec{p} is the time-dependent dipole moment of the distribution.

Also recall that the maxwell Stress Tensor is given by

$$T_{ij} = \varepsilon_o (E_i E_j - \frac{1}{2} \delta_{ij} E^2) + \frac{1}{\mu_o} (B_i B_j - \frac{1}{2} \delta_{ij} B^2)$$



(a) Consider a source of radiation consisting of two tiny balls located at $\pm d/2$ on the z axis. The charges are hooked up to a power source so that charge on the top ball is $q_o \sin \omega t$ whereas the charge on the bottom ball is $-q_o \sin \omega t$. Find an expression for the x, y and z components of \vec{E} and \vec{B} at a time t in terms of the spherical polar coordinates θ, ϕ , and r as well as the constants q_o and ω .

Here $\vec{p} = dq_o \sin \omega t \hat{z}$,

$$\begin{aligned} \frac{d^2 \vec{p}}{dt^2} &= (-\omega^2 dq_o \sin \omega t) \hat{z} \\ \hat{r} \times \hat{z} &= \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} \times \hat{z} = \frac{-x\hat{y} + y\hat{x}}{r} \\ &= \sin \theta (\sin \phi, -\cos \phi, 0) \\ \hat{r} \times (\hat{r} \times \hat{z}) &= \hat{r} (\hat{r} \cdot \hat{z}) - \hat{z} = \frac{z}{r} \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} - \hat{z} \\ &= \sin \theta (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \end{aligned}$$

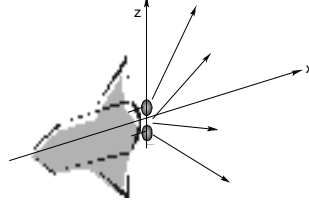
Let us let

$$E_t = -\frac{\mu_o dq_o \omega^2 \sin \omega t}{4\pi r}$$

Then this becomes

$$\begin{aligned}\vec{E} &= E_t \sin \theta (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \\ \vec{B} &= \frac{E_t}{c} \sin \theta (-\sin \phi, \cos \phi, 0)\end{aligned}$$

(b) Assuming 1 and 3 correct, find an expression for $\vec{T}_x = (\overleftrightarrow{T}_{xx}, \overleftrightarrow{T}_{xy}, \overleftrightarrow{T}_{xz})$ and $\vec{T}_x \cdot \hat{r} = \vec{T}_x \cdot (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$.



$$\begin{aligned}\overleftrightarrow{T}_{xj} &= \varepsilon_o(E_x E_j - \frac{1}{2} \delta_{xj} E^2) + \frac{1}{\mu_o}(B_x B_j - \frac{1}{2} \delta_{xj} B^2) \\ &= \varepsilon_o(E_x E_j - \frac{1}{2} \delta_{xj} E_t^2 \sin^2 \theta) + \varepsilon_o(c^2 B_x B_j - \frac{1}{2} \delta_{xj} E_t^2 \sin^2 \theta) \\ &= \varepsilon_o(E_x E_j + c^2 B_x B_j) - \delta_{xj} \varepsilon_o E_t^2 \sin^2 \theta \\ \vec{T}_x &= \varepsilon_o E_t^2 \sin^2 \theta (\cos \theta \cos \phi (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \\ &\quad - \sin \phi (-\sin \phi, \cos \phi, 0) - (1, 0, 0)) \\ &= \varepsilon_o E_t^2 \sin^2 \theta (\cos^2 \theta \cos^2 \phi + \sin^2 \phi - 1, \\ &\quad \cos^2 \theta \cos \phi \sin \phi - \cos \phi \sin \phi, -\cos \theta \sin \theta \cos \phi) \\ &= \varepsilon_o E_t^2 \sin^2 \theta (-\cos^2 \phi \sin^2 \theta, -\sin^2 \theta \cos \phi \sin \phi, -\cos \theta \sin \theta \cos \phi)\end{aligned}$$

Note that $\vec{T}_x \cdot \hat{r}$ has a really simple form:

$$\begin{aligned}\vec{T}_x \cdot \hat{r} &= -\varepsilon_o E_t^2 \sin^2 \theta (-\cos^3 \phi \sin^3 \theta - \sin^3 \theta \cos \phi \sin^2 \phi - \cos^2 \theta \sin \theta \cos \phi) \\ &= \varepsilon_o E_t^2 \sin^2 \theta (-\cos^3 \phi \sin^3 \theta - \sin^3 \theta \cos \phi (1 - \cos^2 \phi) - \cos^2 \theta \sin \theta \cos \phi) \\ &= \varepsilon_o E_t^2 \sin^2 \theta (-\sin^3 \theta \cos \phi - (1 - \sin^2 \theta) \sin \theta \cos \phi) \\ &= \varepsilon_o E_t^2 \sin^2 \theta (-\sin \theta \cos \phi) \\ &= -\varepsilon_o E_t^2 \sin^3 \theta \cos \phi\end{aligned}$$

(c) A new rocket ship is designed to be propelled by expelling photons from a dipole antenna. Half of the radiation from the antenna is allowed to be spewed out the back side of the ship whereas the radiation emitted in the direction of the ship is absorbed as shown in the figure. USE YOUR ANSWER TO PART (b) to find the average force $\langle F_x \rangle$ on the rocket ship. You may assume that the second integral in the expression

$$\frac{d\vec{P}}{dt} = \oint_S \overleftrightarrow{T} \cdot d\vec{a} - \frac{1}{c^2} \frac{d}{dt} \int \vec{S} d\tau$$

will time average to zero.

We will solve

$$\begin{aligned}
 F_x &= \oint_S \vec{T}_x \cdot d\vec{a} \\
 &= r^2 \int_0^\pi \int_{-\pi/2}^{\pi/2} (\vec{T}_x \cdot \hat{r}) \sin \theta d\phi d\theta
 \end{aligned}$$

Note we are technically integrating over the entire sphere, but the absorber is blocking the radiation for $x < 0$, thus this half of the sphere has $\vec{T}_x = 0$. Now let us complete the integral over the hemisphere

$$\begin{aligned}
 F_x &= -\varepsilon_o r^2 E_t^2 \int_0^\pi \int_{-\pi/2}^{\pi/2} (\sin^4 \theta \cos \phi) d\phi d\theta \\
 &= -2\varepsilon_o r^2 E_t^2 \int_0^\pi (\sin^4 \theta) d\theta \\
 &= -\frac{3\pi}{4} \varepsilon_o r^2 E_t^2, \quad E_t = -\frac{\mu_o d q_o \omega^2 \sin \omega t}{4\pi r} \\
 &= -\frac{3}{4} \frac{\mu_o^2 \varepsilon_o^2 d^2 q_o^2 \omega^4 \sin^2 \omega t}{\varepsilon_o 16\pi^2} \\
 \langle F_x \rangle &= -\frac{3}{128} \frac{d^2 q_o^2 \omega^4}{\varepsilon_o \pi^2 c^4}
 \end{aligned}$$

(d) Assuming that the rock ship is traveling at a velocity $v \ll c$ and the power supply moving charge from the ball at $z = d/2$ to the ball at $z = -d/2$ has an internal resistance R , determine an expression for the efficiency of the propulsion method. (i.e. the ratio of power $\langle F_x \rangle v$ going to propell the ship to the average power $\langle P \rangle$ used by the supply.

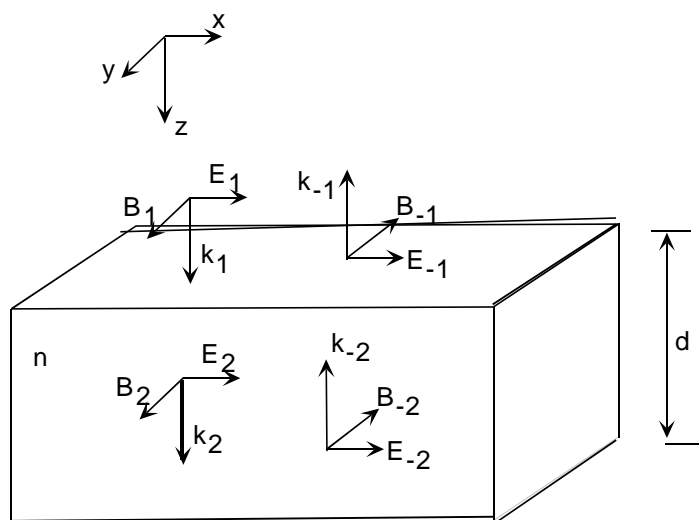
$$P = I^2 R = (q\omega \cos \omega t)^2 R. \quad \langle P \rangle = \frac{1}{2} q^2 \omega^2 R$$

$$\begin{aligned}
 e &= \frac{F_x v}{\langle P \rangle} = \frac{3}{64} \frac{d^2 q_o^2 \omega^4 v}{\varepsilon_o \pi^2 c^4 q^2 \omega^2 R} \\
 &= \frac{3}{64} \frac{(d\omega)^2 v (\frac{1}{\varepsilon_o R})}{c^4}
 \end{aligned}$$

(e) Assuming $R = 1\Omega$ and given $\varepsilon_o = 8.85 \times 10^{-12} \text{ s/m}/\Omega$ and assuming $\omega d \approx v \approx 1000 \text{ ms}^{-1}$, how efficient is the scheme?

$$e = \frac{3}{64} \frac{(10^3 \text{ m/s})^3 (\frac{1}{8.85 \times 10^{-12}} \text{ m/s})}{(3 \times 10^8 \text{ m/s})^4} = 6 \times 10^{-16}$$

Problem 3: Thin window transmission: Etalon that great.



(a) Light of vacuum wavelength $\lambda = 2\pi/k$ travels through vacuum to a flat glass surface (index of refraction n). The light is incident normal to the surface. We eventually will calculate the percent of light transmitted as a function of the thickness d of the plate. To make sure this does not turn into an algebraic nightmare for you, please fill complete the following table in terms of $E_{\pm 1}, E_{\pm 2}, E_3$ and the indices of refraction.

	Electric Field	Magnetic Field
incident #1	$\vec{E}_{+1} = E_{+1}\hat{x}e^{i(kz-\omega t)}$	$\vec{B}_{+1} = \frac{E_{+1}}{c}\hat{y}e^{i(kz-\omega t)}$
reflected #1	$\vec{E}_{-1} = E_{-1}\hat{x}e^{i(-kz-\omega t)}$	$\vec{B}_{-1} = -\frac{E_{-1}}{c}\hat{y}e^{-i(kz-\omega t)}$
transmitted #2	$\vec{E}_{+2} = E_{+2}\hat{x}e^{i(k_2z-\omega t)}$	$\vec{B}_{+2} = \frac{nE_{+2}}{c}\hat{y}e^{i(k_2z-\omega t)}$
reflected #2	$\vec{E}_{-2} = E_{-2}\hat{x}e^{i(-k_2z-\omega t)}$	$\vec{B}_{-2} = \frac{-nE_{-2}}{c}\hat{y}e^{i(-k_2z-\omega t)}$
transmitted #3	$\vec{E}_3 = E_3\hat{x}e^{i(kz-\omega t)}$	$\vec{B}_3 = \frac{E_3}{c}\hat{y}e^{i(kz-\omega t)}$

(I have taken the liberty of using the fact that $k_3 = k_1 = k$.)

(b) The static boundary value conditions

$$\begin{aligned} E_1^{\parallel} &= E_2^{\parallel} & \varepsilon_1 E_1^{\perp} &= \varepsilon_2 E_2^{\perp} \\ B_1^{\parallel}/\mu_1 &= B_2^{\parallel}/\mu_2 & B_1^{\perp} &= B_2^{\perp} \end{aligned}$$

accurately determine the ratio of reflected to transmitted light. For the study here, we may assume that $\mu_1 = \mu_2 = \mu_3$, making the two relevant boundary value conditions simple. Find \vec{E}_3 in terms of \vec{E}_1 .

Only the parallel boundary conditions are relevant, leading to

$$\begin{aligned} E_{+1} + E_{-1} &= E_{+2} + E_{-2} \\ E_{+2}e^{ik_2d} + E_{-2}e^{-ik_2d} &= E_3e^{ikd} \\ E_{+1} - E_{-1} &= n(E_{+2} - E_{-2}) \\ n(E_{+2}e^{ik_2d} - E_{-2}e^{-k_2d}) &= E_3e^{ikd} \end{aligned}$$

Adding n times the second equation to the last equation gives:

$$\begin{aligned} (1+n)E_3e^{ikd} &= 2nE_{+2}e^{ik_2d} \\ E_{+2} &= \frac{1+n}{2n}E_3e^{i(k-k_2)d} \end{aligned}$$

Adding the first and the third equation gives

$$\begin{aligned} 2E_{+1} &= (1+n)E_{+2} + (1-n)E_{-2} \\ E_{-2} &= \frac{2}{1-n}E_{+1} - \frac{1+n}{1-n}E_{+2} \\ &= \frac{2}{1-n}E_{+1} - \frac{(1+n)^2}{2n(1-n)}E_3e^{i(k-k_2)d} \end{aligned}$$

Plugging this back into the second equation gives

$$\begin{aligned} \frac{1+n}{2n}E_3e^{ikd} + \left(\frac{2}{1-n}E_{+1} - \frac{(1+n)^2}{2n(1-n)}E_3e^{i(k-k_2)d} \right) e^{-ik_2d} &= E_3e^{ikd} \\ \frac{2}{1-n}E_{+1}e^{-k_2d} &= E_3e^{ikd} \left(1 - \frac{1+n}{2n} + \frac{(1+n)^2}{2n(1-n)}e^{-i2k_2d} \right) \\ 4nE_{+1}e^{-i(k_2+k)d} &= E_3(2n(1-n) - (1+n)(1-n) + (1+n)^2e^{-i2k_2d}) \\ E_3 &= \frac{4n}{(1+n)^2e^{-i2k_2d} - n^2 + 2n - 1}E_{+1}e^{-i(k_2+k)d} \\ &= \frac{4n}{(n+1)^2 - (n-1)^2e^{i2k_2d}}E_{+1}e^{i(k_2-k)d} \end{aligned}$$

(c) Find the reflection coefficient $R = [\text{Power reflected} : \text{Power in}]$ as a function of $\alpha = k_2d$ and n .

To determine this, we note that

$$\begin{aligned} (n+1)^2 - (n-1)^2e^{i2k_2d} &= (n+1)^2 - (n-1)^2 \cos(2k_2d) + i(n-1)^2 \sin(2k_2d) \\ &= ((n+1)^4 + (n-1)^4 - 2(n+1)^2(n-1)^2 \cos(2k_2d))^{1/2} \exp(i\phi) \end{aligned}$$

Here ϕ is an unimportant angle. The real part of E_3 is given by

$$\text{Re}[E_3] = \frac{4n}{((n+1)^4 + (n-1)^4 - 2(n+1)^2(n-1)^2 \cos(2k_2d))^{1/2}} E_{+1} \cos((k_2-k)d - \omega t + \phi)$$

The power transmitted is proportional to the time average of $\text{Re}[E_3]^2$ and the transmission is

$$\begin{aligned}
 T &= \frac{\langle \text{Re}[E_3]^2 \rangle}{\frac{1}{2}E_{+1}^2} \\
 &= \left(\frac{4n}{((n+1)^4 + (n-1)^4 - 2(n+1)^2(n-1)^2 \cos(2kd))^{1/2}} \right)^2 \\
 R &= 1 - T = 1 - \frac{16n^2}{((n+1)^4 + (n-1)^4 - 2(n+1)^2(n-1)^2 \cos(2k_2d))} \\
 &= \frac{4 \sin^2(k_2d)}{(\beta^2 + \frac{1}{\beta^2} - 2 + 4 \sin^2(k_2d))} \quad \text{with } \beta = \frac{n+1}{n-1}
 \end{aligned}$$

Of course on a test you may not get to such a pretty final result. For large β (n not too different from 1) this becomes

$$R \approx 4 \left[\frac{n-1}{n+1} \right]^2 \sin^2(k_2d)$$

Problem 4: Mutual Inductance

Use the fact that the integral

$$\frac{1}{2} \int_0^{2\pi} \frac{\cos \phi}{(1 - \varepsilon \cos^2 \frac{\phi}{2})^{1/2}} d\phi = \pi {}_2F_1(\varepsilon) - 2K(\varepsilon) \quad (\text{for } 0 < \varepsilon < 1)$$

to find an expression for the mutual inductance M_{ab} between two concentric wire loops, one of radius a and the other of radius b . Show explicitly that

$$M_{ab} = M_{ba}.$$

We start by finding the magnetic field from loop a. We have

$$\begin{aligned} \vec{B}_a &= \vec{\nabla} \times \vec{A} \\ \vec{A} &= \frac{\mu_o I}{4\pi} \int \frac{d\vec{\ell}_a}{|\vec{r} - \vec{r}_a|} \end{aligned}$$

The flux through loop b is given by

$$\begin{aligned} \Phi_b &= \int \int_{\text{closed}} \vec{B}_a \cdot d\vec{a}_b \\ &= \int \int_{\text{closed}} \vec{\nabla} \times \vec{A} \cdot d\vec{a}_b \\ &= \int \vec{A} \cdot d\vec{\ell}_b \\ &= \frac{\mu_o I}{4\pi} \int \int \frac{d\vec{\ell}_a \cdot d\vec{\ell}_b}{|\vec{r}_a - \vec{r}_b|} \end{aligned}$$

So

$$M_{ab} = \frac{\mu_o}{4\pi} \int \int \frac{d\vec{\ell}_a \cdot d\vec{\ell}_b}{|\vec{r}_a - \vec{r}_b|}$$

letting $\vec{r}_b = b(\cos \phi_b, \sin \phi_b, 0)$, $\vec{r}_a = a(\cos \phi_a, \sin \phi_a, 0)$

$$\begin{aligned}
|\vec{r}_b - \vec{r}_a| &= ((b \cos \phi_b - a \cos \phi_a)^2 + (b \sin \phi_b - a \sin \phi_a)^2)^{1/2} \\
&= (a^2 + b^2 - 2ab \cos(\phi_a - \phi_b))^{1/2} \\
d\vec{\ell}_a &= \frac{d\vec{r}_a}{d\phi_a} d\phi_a = a(-\sin \phi_a, \cos \phi_a, 0) d\phi_a \\
d\vec{\ell}_b &= \frac{d\vec{r}_b}{d\phi_b} d\phi_b = b(-\sin \phi_b, \cos \phi_b, 0) d\phi_b \\
d\vec{\ell}_a \cdot d\vec{\ell}_b &= ab \cos(\phi_a - \phi_b) d\phi_a d\phi_b \\
M_{ab} &= \frac{\mu_o}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{ab \cos(\phi_a - \phi_b) d\phi_a d\phi_b}{(a^2 + b^2 - 2ab \cos(\phi_a - \phi_b))^{1/2}} \\
&= \frac{\mu_o}{2} \int_0^{2\pi} \frac{ab \cos(\phi) d\phi}{(a^2 + b^2 - 2ab \cos(\phi))^{1/2}} \\
&= \frac{\mu_o}{2} \int_0^{2\pi} \frac{ab \cos(\phi) d\phi}{(a^2 + b^2 + 2ab - 2ab(1 + \cos \phi))^{1/2}} \\
&= \frac{\mu_o}{2} \int_0^{2\pi} \frac{ab \cos(\phi) d\phi}{(a^2 + b^2 + 2ab - 4ab \cos^2 \frac{\phi}{2})^{1/2}} \\
&= \frac{\mu_o}{2} \int_0^{2\pi} \frac{ab \cos(\phi) d\phi}{((a+b)^2 - 4ab \cos^2 \frac{\phi}{2})^{1/2}} \\
&= \frac{\mu_o}{2} \frac{ab}{a+b} \int_0^{2\pi} \frac{\cos(\phi) d\phi}{(1 - \varepsilon \cos^2 \frac{\phi}{2})^{1/2}} \quad \varepsilon = \frac{4ab}{(a+b)^2} \\
&= \mu_o \frac{ab}{a+b} (\pi {}_2F_1(\frac{4ab}{(a+b)^2}) - 2K(\frac{4ab}{(a+b)^2})) \\
&= M_{ba}
\end{aligned}$$