1 Unit #5: Radiation from continuous charge distributions

Radiation is the escape of electromagnetic radiation from a charge distribution. More formally, to define radiation we first imagine a localized charge distribution. We then write down the energy escaping a sphere of radius r surrounding the distribution per unit time:

$$P(r) = \oint_r \vec{S} \cdot d\vec{a} \tag{1}$$

$$= \frac{1}{\mu_o} \oint_r (\vec{E} \times \vec{B}) \cdot d\vec{a} \tag{2}$$

When we take the limit as $r \to \infty$ we find the energy that actually escapes the charge distribution, never to return.

$$P_{rad} = \lim_{r \to \infty} P(r) \tag{3}$$

The first thing we note is that a static field does not radiate power. To see this, recall that the multipole expansion of a static field far away from a steady-state charge distribution:

$$\vec{E} = \frac{1}{4\pi\varepsilon_o} \left(\frac{Q\hat{r}}{r^2} + \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} + ... \right) \tag{4}$$

and the magnetic field far away from a steady-state charge distribution:

$$\vec{B} = \left(\frac{\mu_o}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} + ... \right) \tag{5}$$

We see that the lowest-order term in $\vec{E} \times \vec{B}$ is of order $\frac{1}{r^5}$. On the other hand the area of the integral goes up as r^2. Thus the power slipping by a sphere surrounding by a steady-state localized charge distribution is going to drop as $1/r^3$ or faster. Apparently, to produce radiated power, we need to leave steady-state conditions. (Even if magnetic monopoles were to exist, the Poynting vector would drop like $1/r^4$.)

1.1 Radiation from a simple electric Dipole

1.1.1 The simplest dipole radiator

We first consider a simple case of two charges $+q(t)$ and $-q(t)$ separated by a distance d located at $\pm d/2$ on the z axis. To keep the calculation of the retarded time simple, we imagine that the location of the charge is fixed but the charge varies with time according to

$$q(t) = q_o \cos(\omega t) \tag{6}$$

The charge varies by a straight wire that connects the two points and carries a current $I = dq/dt$.

1.1.2 The scalar potential

The retarded scalar potential is given by

$$V = \frac{q_o}{4\pi\varepsilon_o}\left(\frac{\cos(\omega(t - |\vec{r} - \frac{d}{2}\hat{z}|/c)}{|\vec{r} - \frac{d}{2}\hat{z}|} + \frac{\cos(\omega(t - |\vec{r} + \frac{d}{2}\hat{z}|/c)}{|\vec{r} + \frac{d}{2}\hat{z}|}\right) \tag{7}$$

We will be interested in both $\vec{E}$ and $\vec{B}$ far a way from the charges. We therefore can assume that $r >> d$ so that

$$|\vec{r} - \frac{d}{2}\hat{z}| = \sqrt{r^2 - \vec{d}\cdot\vec{r} + d^2/4} \tag{8}$$

$$\approx r - \frac{1}{2}d\cos\theta \tag{9}$$

With this and similar approximations for $|\vec{r} + \frac{d}{2}\hat{z}|$ we have

$$V \approx \frac{q_o}{4\pi\varepsilon_o}\left(\frac{\cos(\omega(t - r/c + d\cos\theta/2c)}{r - d\cos\theta/2} - \frac{\cos(\omega(t - r/c - d\cos\theta/2c)}{r + d\cos\theta/2}\right) \tag{10}$$

$$= \frac{q_o}{4\pi\varepsilon_o}\left(\frac{\cos(\omega(t - r/c + \delta))}{r - \delta c/\omega} - \frac{\cos(\omega(t - r/c - \delta))}{r + \delta c/\omega}\right) \tag{11}$$

Here $\delta = \omega d\cos\theta/2c$.

1.1.3 The dipole approximation

Now we assume that $d\omega/c << 0$ (or that $d << \lambda$). This may not be a great approximation for a radio antenna (of length comparable to the transmitted wavelength) but it is very often justified approximation for atomic and molecular radiation. For example, an atom only a few angstroms across will radiate visible light from $4000 - 7000$ angstroms. With the approximation $d << \lambda$, we may make a Taylor series approximation about δ:

$$V \approx \frac{q_o}{4\pi\varepsilon_o}\left(\begin{array}{c} 0\,\delta^0 + \left(\frac{-r\sin(\omega(t-r/c))+c\cos(\omega(t-r/c))c}{r^2}\right)\delta^1 \\ + \left(\frac{-r\sin(\omega(t-r/c))+c\cos(\omega(t-r/c))}{r^2}\right)\delta^1 \\ +O(\delta^3) \end{array}\right) \tag{12}$$

Notice that we did not have to work very hard to see that the δ^0 and δ^3 terms of the Taylor series vanish: The function V is even in δ. Continuing with the simplification and letting $p_o = q_o d$ we have

$$V \approx \frac{p_o\cos\theta}{4\pi\varepsilon_o}\left(\frac{\cos(\omega(t - r/c))}{r^2} - \frac{\omega\sin(\omega(t - r/c))}{cr}\right) \tag{13}$$

$$= \frac{p_o\cos\theta}{4\pi\varepsilon_o}\frac{\omega}{cr}\left(\frac{c}{r\omega}\cos(\omega(t - r/c)) - \sin(\omega(t - r/c))\right) \tag{14}$$

Now we remember we are ultimately interested in the Poynting vector in the limit of large r. We suppose that we will look far enough away so that the term dropping as r^2 vanishes. Actually this is not a big new approximation. We are really saying $r >> c/\omega$. But we have already said $d << c/\omega$. Thus this new approximation is valid if $r >> d$, which we have already stated. In any case, we now have

$$V \approx \frac{-p_o \cos\theta}{4\pi\varepsilon_o}\frac{\omega}{cr}\sin(\omega(t - r/c)) \tag{15}$$

This is the approximate form of the potential we have desired.

1.1.4 The vector potential

Now we consider the vector potential $\vec{A}$. Fortunately, the lowest order term in d is much easier to find than it was for the case of finding V:

$$\begin{aligned}
\vec{A} &= \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}', t_r)d\tau}{|\vec{r} - \vec{r}'|} \quad \text{(in general)} && (16)\\
&= \frac{\mu_o}{4\pi}\int \frac{I(\vec{r}', t_r)d\vec{\ell}}{|\vec{r} - \vec{r}'|} \quad \text{(for a wire)} && (17)\\
&= \frac{-\mu_o\widehat{z}}{4\pi}\int_{-d/2}^{d/2} \frac{\omega q_o \sin(\omega(t - |\vec{r} - \vec{r}'|/c)dz}{|\vec{r} - \vec{r}'|} && (18)\\
&\approx \frac{-\mu_o\widehat{z}}{4\pi}\int_{-d/2}^{d/2} \frac{\omega \sin(\omega(t - r/c + d\omega\cos\theta/2c))}{r - d\omega\cos\theta/2c}dz && (19)
\end{aligned}$$

Again we take the lowest term that doesn't vanish in d. In this case we need only take the first term in the Taylor series:

$$\begin{aligned}
\vec{A} &\approx \frac{-\mu_o p_o \omega \sin(\omega(t - r/c))}{4\pi r}\widehat{z} && (20)\\
&= \frac{-\mu_o p_o \omega \sin(\omega(t - r/c))}{4\pi r}\left[\cos\theta\widehat{r} - \sin\theta\widehat{\theta}\right] && (21)
\end{aligned}$$

1.1.5 Finding $\vec{E}$ and $\vec{B}$

Now we compute the fields. To get these we need $\vec{\nabla}V$, $\vec{\nabla}\times\vec{A}$ and $\partial\vec{A}/\partial t$:

$$\begin{aligned}
\vec{\nabla}V &= \frac{\partial V}{\partial r}\widehat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\widehat{\theta} && (22)\\
&= \frac{p_o \cos\theta}{4\pi\varepsilon_o r}\frac{\omega}{c}\left[\frac{\sin(\omega(t-r/c))}{r} + \left(\frac{\omega}{c}\right)\cos(\omega(t-r/c))\right]\widehat{r} && (23)\\
&\quad + \frac{p_o \sin\theta}{4\pi\varepsilon_o r^2}\frac{\omega}{c}\sin(\omega(t-r/c))\widehat{\theta} && (24)\\
&\approx \frac{p_o \cos\theta}{4\pi\varepsilon_o r}\left(\frac{\omega}{c}\right)^2\cos(\omega(t-r/c))\widehat{r} && (25)\\
&= \frac{\mu_o p_o \omega^2 \cos\theta}{4\pi r}\cos(\omega(t-r/c))\widehat{r} && (26)\\
\frac{\partial\vec{A}}{\partial t} &= \frac{-\mu_o p_o \omega^2 \cos(\omega(t-r/c))}{4\pi r}\widehat{z} && (27)\\
&= \frac{-\mu_o p_o \omega^2 \cos(\omega(t-r/c))}{4\pi r}\left[\cos\theta\widehat{r} - \sin\theta\widehat{\theta}\right] && (28)\\
\vec{E} &= -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} && (29)\\
&= \frac{-\mu_o p_o \omega^2 \cos(\omega(t-r/c))}{4\pi r}\sin\theta\widehat{\theta} && (30)
\end{aligned}$$

Now we find the curl of $\vec{A}$ in order to obtain $\vec{B}$:

$$\begin{aligned}
\vec{B} &= \vec{\nabla}\times\vec{A} && (31)\\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right]\widehat{\phi} + && (32)\\
&\quad \text{terms that are zero by inspection} && (33)\\
&= \left[\frac{-\mu_o p_o \omega^2 \sin\theta\cos(\omega(t-r/c))}{4\pi c r} + \frac{\mu_o p_o \sin\theta\sin(\omega(t-r/c))}{4\pi r^2}\right]\widehat{\phi} && (34)\\
&\approx \frac{-\mu_o p_o \omega^2 \sin\theta\cos(\omega(t-r/c))}{4\pi c r}\widehat{\phi} && (35)
\end{aligned}$$

1.1.6 The Poynting vector

Finally, we may determine the Poynting vector for the oscillating dipole and the power radiated:

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu_o}\vec{E}\times\vec{B} && (36)\\
&= \frac{\mu_o p_o^2 \omega^4}{16\pi^2 c r^2}\sin^2\theta\cos^2(\omega(t-r/c))\widehat{r} && (37)\\
&= \left[\frac{\pi^2 q_o^2}{\varepsilon_o r^2}\right]\frac{d^2 c}{\lambda^4}\sin^2\theta\cos^2(\omega(t-r/c))\widehat{r} && (38)
\end{aligned}$$

The last line has been written down using $\lambda = 2\pi c/\omega$ simply as a unit check: The term in square brackets has units of force whereas the rest has units of distance per area per time. Thus $\vec{S}$ has units of energy per area per unit time as it should.

For many measurements of electromagnetic radiation, we measure the average power over many cycles. In this case, the time averaged flux becomes

$$\frac{dP_{dipole}}{d\Omega} = \lim_{r\to\infty}(\left\langle \vec{S}_{dipole} \cdot \hat{r} \right\rangle r^2) = \frac{\mu_o p_o^2 \omega^4}{32\pi^2 c} \sin^2\theta \tag{39}$$

1.1.7 Power Radiated and why the sky is blue

Notice the flux is radial and drops off as $1/r^2$. This is the drop off required to have a non-zero amount of radiated power:

$$\begin{aligned} \langle P_{dipole} \rangle &= \oint \left\langle \vec{S} \right\rangle \cdot d\vec{a} &(40)\\ &= \oint \frac{dP_{dipole}}{d\Omega} d\Omega &(41)\\ &= \frac{\mu_o \omega^4}{12\pi c} p_o^2 &(42) \end{aligned}$$

The ω^4 dependence of the power radiated by an oscillating dipole is the reason the sky is blue: White sunlight passing through our atmosphere causes charges to oscillate. The oscillating charges radiate more in the blue than in the red, causing our sky to be blue.

1.2 A simple Magnetic Dipole Radiator

We imagine a loop of radius b in the x-y axis with no net charge, but carrying a current

$$I(t) = I_o \cos(\omega t) \tag{43}$$

The magnetic dipole is given by

$$\begin{aligned} \vec{m} &= \pi b^2 I_o \widehat{z} \cos\omega t &(44)\\ &= m_o \widehat{z} \cos\omega t &(45) \end{aligned}$$

$$\vec{A} = \frac{\mu_o}{4\pi} \oint \frac{I_o \cos(\omega(t - |\vec{r} - \vec{r}'|/c))}{|\vec{r} - \vec{r}'|} d\vec{\ell}' \tag{46}$$

Now $\vec{A}$ can have $\widehat{\phi}$ but, by symmetry, it can not have a $\widehat{s}$ or $\widehat{z}$ component. We can solve the integral by writing:

$$\begin{aligned} \vec{r}' &= b(\cos\phi\widehat{x} + \sin\phi\widehat{y}) &(47)\\ d\vec{\ell}' &= b(-\sin\phi\widehat{x} + \cos\phi\widehat{y})d\phi &(48)\\ \vec{r} &= (\sin\theta, 0, \cos\theta)r &(49)\\ |\vec{r} - \vec{r}'|^2 &= ((r\sin\theta - b\cos\phi)^2 + b^2\sin\phi^2 + r^2\cos\theta^2) &(50)\\ &= r^2 + b^2 - 2br\cos\phi\sin\theta &(51) \end{aligned}$$

Here we have assumed that $\vec{r}$ is on the x axis. This will result in a y-component of $\vec{A}$. By symmetry of the problem, the final answer will be true in general if we take the final result and replace $\widehat{y}$ with $\widehat{\phi}$.

Continuing with the substitutions, we have

$$\vec{A} = \frac{\mu_o b}{4\pi} \oint \frac{I_o \cos(\omega(t - \sqrt{r^2 + b^2 - 2br\cos\phi\sin\theta}/c))}{\sqrt{r^2 + b^2 - 2br\cos\phi\sin\theta}} (-\sin\phi\widehat{x} + \cos\phi\widehat{y}) d\phi \tag{52}$$

Notice that we have not yet made any approximations. Notice also that the $\widehat{x}$ component of this integral is odd in ϕ and vanishes as expected. We are left with

$$\vec{A} = \frac{\mu_o b I_o}{4\pi} \widehat{y} \oint \frac{I_o \cos\phi \cos(\omega(t - \sqrt{r^2 + b^2 - 2br\cos\phi\sin\theta}/c))}{\sqrt{r^2 + b^2 - 2br\cos\phi\sin\theta}} d\phi \tag{53}$$

(on the x axis) (54)

$$= \frac{\mu_o b I_o}{4\pi} \widehat{\phi} \oint \frac{\cos\phi \cos(\omega(t - \sqrt{r^2 + b^2 - 2br\cos\phi\sin\theta}/c))}{\sqrt{r^2 + b^2 - 2br\cos\phi\sin\theta}} d\phi \tag{55}$$

Again we are interested in the case of very large r. For this case, the b^2 term can be ignored and the radical term can be simplified:

$$\vec{A} = \frac{\mu_o b I_o}{4\pi} \widehat{\phi} \oint \frac{\cos\phi \cos(\omega(t - \frac{r - b\cos\phi\sin\theta}{c}))}{r - b\cos\phi\sin\theta} d\phi \tag{56}$$

Now I assume that $b << \lambda = 2\pi c/\omega$. To find the implication of this assumptions, I let $\delta = \omega b \cos\phi \sin\theta / c$. The integrand becomes

$$INT = \cos\phi \frac{\cos(\omega(t - r/c) + \delta)}{r - \delta c/\omega} \tag{57}$$

Now I can Taylor series this in terms of δ

$$INT = \cos\phi(\frac{\cos\omega(t - rc)}{r} + \frac{-r\sin(\omega(t - r/c)) + (c/\omega)\cos\omega(t - r/c)}{r^2}\delta + \tag{58}$$

$$= \cos\phi \frac{\cos\omega(t - rc)}{r} + \tag{59}$$

$$\frac{(-r\sin(\omega(t - r/c)) + (c/\omega)\cos\omega(t - r/c))}{r^2} \frac{\omega b \sin\theta}{c} \cos^2\phi + ... \tag{60}$$

The first term integrates to zero whereas the second term integrates to the result

$$\vec{A} = \frac{\mu_o \pi b^2 I_o}{4\pi} (\frac{\omega}{c}) \frac{\sin\theta}{r} \sin(\omega t - \omega r/c) \widehat{\phi} \tag{61}$$

$$= -\frac{\mu_o m_o \omega}{4\pi c} \frac{\sin\theta}{r} \sin(\omega t - \omega r/c) \widehat{\phi} \tag{62}$$

The electric field is given by

$$\begin{aligned}
\vec{E} &= -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_o m_o \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos(\omega t - \omega r/c)\hat{\phi} \\
\vec{B} &= \vec{\nabla}\times\vec{A} = \frac{1}{r}\left[\frac{1}{\sin\theta}\hat{r}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \hat{\theta}\frac{\partial}{\partial r}(rA_\phi)\right] \\
&= -\frac{\mu_o m_o \omega}{4\pi c r}\left[\frac{2\cos\theta}{r}\hat{r}\sin(\omega t - \omega r/c) + \frac{\omega\sin\theta}{c}\cos(\omega t - \omega r/c)\hat{\theta}\right] \\
&= -\frac{\mu_o m_o \omega^2}{4\pi c^2}\frac{\sin\theta}{r}\hat{\theta} + O[\frac{1}{r^2}] \\
\vec{S} &= \frac{1}{\mu_o}\vec{E}\times\vec{B} = \frac{\mu_o m_o \omega^4}{16\pi^2 c^3}m_o^2\frac{\sin^2\theta}{r^2}
\end{aligned}$$

$$\frac{d < P_{mag} >}{d\Omega} = \lim_{r\to\infty}(\left\langle \vec{S}_{mag}\cdot\hat{r}\right\rangle r^2) = \frac{\mu_o\omega^4}{32\pi^2 c}\left(\frac{m_o}{c}\right)^2\sin^2\theta \quad (63)$$

$$\frac{d < P_{dipole} >}{d\Omega} = \lim_{r\to\infty}(\left\langle \vec{S}_{dipole}\cdot\hat{r}\right\rangle r^2) = \frac{\mu_o\omega^4}{32\pi^2 c}(p_o^2)\sin^2\theta \quad (64)$$

Here we have placed the dipole result nearby for comparison.

2 Radiation in the general case

Now let us consider the potential

$$V = \frac{1}{4\pi\varepsilon_o}\int\frac{1}{|\vec{r}-\vec{r}'|}\rho(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})d\tau' \quad (65)$$

Again let us assume we are looking far from the region in space with $\rho \neq 0$. In this case we have $r >> r'$ so that

$$\begin{aligned}
|\vec{r}-\vec{r}'| &= (r^2 + r'^2 - 2rr'\cos\theta)^{1/2} \\
&= r - r'\cos\theta + O[(\frac{r'}{r})^2] \quad (66) \\
\frac{1}{|\vec{r}-\vec{r}'|} &= \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta_{r,r'}) \\
&= \frac{1}{r} + \frac{r'}{r^2}\cos\theta + O[(\frac{r'}{r})^2] \quad (67)
\end{aligned}$$

Ignoring the terms of order r'/r, we have

$$\frac{1}{|\vec{r}-\vec{r'}|} \approx \frac{1}{r} - \frac{\vec{r'}\cdot\vec{r}}{r^3} \tag{68}$$

$$|\vec{r}-\vec{r'}| \approx r - \frac{\vec{r'}\cdot\vec{r}}{r} \tag{69}$$

$$\approx \rho(\vec{r'}, t - \frac{r}{c} + \frac{\vec{r'}\cdot\vec{r}}{rc}) \tag{70}$$

$$\approx \rho(\vec{r'}, t - \frac{r}{c}) + \dot{\rho}(\vec{r'}, t - \frac{r}{c})\frac{\vec{r'}\cdot\vec{r}}{rc} + \frac{1}{2}\ddot{\rho}(\vec{r'}, t - \frac{r}{c})(\frac{\vec{r'}\cdot\vec{r}}{rc})^2 + \cdots \tag{71}$$

Up to now we have made no real approximation. We have only claimed that we are looking far away. Now we make the dipole approximation: That is we assume that the higher order derivatives are not too big. Remember we made a similar assumption with the simple dipole radiator when we assumed $d << \lambda = 2\pi c/\omega$. In that case the nth derivative of ρ was proportional to ω^n and we needed $(\frac{\omega\vec{r'}\cdot\hat{r}}{c})^n$ not to be too big. With the dipole approximation, we can say

$$\rho(\vec{r'}, t - \frac{|\vec{r}-\vec{r'}|}{c}) \approx \rho(\vec{r'}, t - \frac{r}{c}) + \dot{\rho}(\vec{r'}, t - \frac{r}{c})\frac{\vec{r'}\cdot\vec{r}}{rc} \tag{72}$$

and

$$V \approx \frac{1}{4\pi\varepsilon_o}\int\left[\frac{1}{r} - \frac{\vec{r'}\cdot\vec{r}}{r^3}\right]\left(\rho(\vec{r'}, t - \frac{r}{c}) + \dot{\rho}(\vec{r'}, t - \frac{r}{c})\frac{\vec{r'}\cdot\vec{r}}{rc}\right)d\tau' \tag{73}$$

$$= \frac{1}{4\pi\varepsilon_o r}\left[Q + \frac{\hat{r}\cdot\vec{p}}{r} + \frac{1}{c}\hat{r}\cdot\dot{\vec{p}}\right] + O(\frac{1}{r^3}) \tag{74}$$

$$V_{dipole} = \frac{\mu_o c}{4\pi r}\hat{r}\cdot\dot{\vec{p}} \tag{75}$$

The vector potential is given by

$$\vec{A} \approx \frac{\mu_o}{4\pi}\int\left[\frac{1}{r} - \frac{\vec{r'}\cdot\vec{r}}{r^3}\right]\left(\vec{J}(\vec{r'}, t - \frac{r}{c}) + \dot{\vec{J}}(\vec{r'}, t - \frac{r}{c})\frac{\vec{r'}\cdot\vec{r}}{rc}\right)d\tau' \tag{76}$$

$$= \frac{\mu_o}{4\pi}\left[\frac{1}{r}\int \vec{J}(\vec{r'}, t - \frac{r}{c})d\tau' + \frac{1}{rc}\int \dot{\vec{J}}(\vec{r'}, t - \frac{r}{c})(\vec{r'}\cdot\hat{r})d\tau'\right] + O(\frac{1}{r^3}) \tag{77}$$

Let us compare the two integrals that contribute to $\vec{A}$. (Here your book is wrong. On page 456 it claims that only the first integral contributes to the $1/r$ term.) The second integral falls as $1/r$, but it contains an extra derivative in time and an added factor of r'. In short, the second integral is smaller than the first by a factor of $r'\omega/c \approx d/\lambda$ where λ is the characteristic wavelength of the radiation and d is the size of the radiator. Thus only the first term contributes in the dipole approximation. To simplify this first integral, we note that

$$\frac{\partial p_x}{\partial t} = \frac{\partial}{\partial t} \int\int\int x\rho' d\tau' \tag{78}$$

But, by conservation of charge, $\vec{\nabla} \cdot \vec{J} = -\partial\rho/\partial t$ so that

$$\frac{\partial p_x}{\partial t} = -\int\int\int x'\vec{\nabla}' \cdot \vec{J}' d\tau' \tag{79}$$

Now let use use that fact that

$$\vec{\nabla} \cdot (f\vec{C}) = (\vec{\nabla} f) \cdot \vec{C} + f\vec{\nabla} \cdot \vec{C} \tag{80}$$

to write

$$x'\vec{\nabla}' \cdot \vec{J}' = \vec{\nabla} \cdot (x'\vec{J}') - J'_x \tag{81}$$

leading to

$$\begin{aligned} \frac{\partial p_x}{\partial t} &= -\int\int\int \vec{\nabla} \cdot (x'\vec{J}') d\tau' + \int\int\int J_x d\tau' & (82) \\ &= -\int\int (x'\vec{J}') \cdot d\vec{a}' + \int\int\int J'_x d\tau' & (83) \end{aligned}$$

But the charge distribution is localized, so $\vec{J}'$ is zero at the outside surface. We have thus shown that

$$\frac{\partial p_x}{\partial t} = \int J'_x d\tau' \tag{84}$$

It follows that

$$\frac{\partial \vec{p}}{\partial t} = \int \vec{J}' d\tau' \tag{85}$$

Thus $\vec{A}$ is given by

$$\begin{aligned} \vec{A}_{dipole} &= \frac{\mu_o}{4\pi r} \dot{\vec{p}} & (86) \\ V_{dipole} &= \frac{\mu_o c}{4\pi r} \hat{r} \cdot \dot{\vec{p}} & (87) \end{aligned}$$

Here we have ignored the second term (part of the dipole approximation.)

Now we evaluate the $\vec{E}$ and $\vec{B}$ fields:

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \tag{88}$$

$$= -\frac{\mu_o c}{4\pi}[(-\frac{1}{r^2}\hat{r}\cdot\dot{\vec{p}} - \frac{1}{rc}\hat{r}\cdot\ddot{\vec{p}})\hat{r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}(\hat{r}\cdot\dot{\vec{p}})] - \frac{\mu_o}{4\pi r}\ddot{\vec{p}} \tag{89}$$

$$= \frac{\mu_o}{4\pi r}((\hat{r}\cdot\ddot{\vec{p}})\hat{r} - \ddot{\vec{p}}) + O[r^{-2}] \tag{90}$$

$$= \frac{\mu_o}{4\pi r}\hat{r}\times(\hat{r}\times\ddot{\vec{p}}) \tag{91}$$

$$E = \frac{\mu_o}{4\pi r}\left|\ddot{\vec{p}}\right|\sin\theta \tag{92}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_o}{4\pi}((\vec{\nabla}\frac{1}{r})\times\dot{\vec{p}} + \frac{\mu_o}{4\pi r}\vec{\nabla}\times\dot{\vec{p}} \tag{93}$$

$$= \frac{\mu_o}{4\pi rc}\ddot{\vec{p}}\times\hat{r} \tag{94}$$

$$B = \frac{\mu_o}{4\pi rc}\left|\ddot{\vec{p}}\right|\sin\theta \tag{95}$$

$$\vec{S} = \frac{1}{\mu_o}\vec{E}\times\vec{B} = \frac{\mu_o}{16\pi^2 r^2 c}\left|\ddot{\vec{p}}\right|^2\sin^2\theta \tag{96}$$

$$\frac{dP_{dipole}}{d\Omega} = \lim_{r\to\infty}(\vec{S}_{dipole}\cdot\hat{r}r^2) = \frac{\mu_o}{16\pi^2 c}\left|\ddot{\vec{p}}\right|^2\sin^2\theta \tag{97}$$

The total radiated energy is then given by

$$P_{dipole} = \oint\frac{dP_{dipole}}{d\Omega}d\Omega \tag{98}$$

$$= \frac{\mu_o}{6\pi^2 c}\left|\ddot{\vec{p}}\right|^2 \tag{99}$$

Note this differs by a factor of two from the previous result simply because the previous result was time averaged whereas this is the instantaneous power radiated.