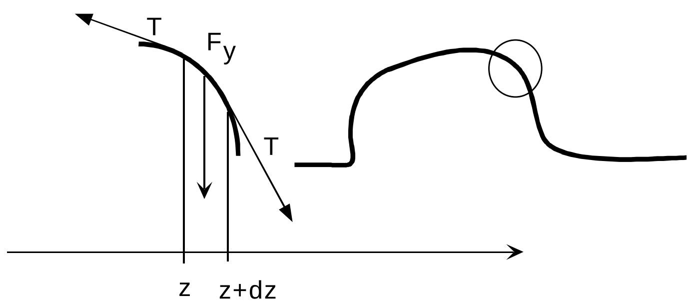
1 Lecture Notes Unit 3: RADIATION

1.1 Review of the wave equation

To review the physics of the wave equation, we return to mechanics simple because most of our intuition is more grounded here than in electrodymanics. To start, we ask ourselves, what is a wave? The answer is of course the propagation of a disturbance through a media. For the case of electromagnetic radiation, this is the propagation of the magnitude of the electric and magnet field. For the case of a mechanical system such as a rope streched between to posts, this is the propagation of the position of an element of the matter from equilibrium. We know if such a rope is plucked, the disturbance runs down the rope (without, of course any mass moving down the wire.) Can we explain the phenomena with a simple model?- you bet.



(1)

Let us consider the forces acting on a small segement of rope spanning the distance from z to $z + \Delta z$. The total force on this rope is given by

$$\Delta \vec{F} = \vec{T}_z + \vec{T}_{z+dz} \tag{2}$$

At $z + \Delta z$ the direction of the tension follows the direction of the rope, which is in turn given by the vector

$$\Delta \hat{l} = \left[\frac{(\frac{\partial y}{\partial z}, 1)}{((\frac{\partial y}{\partial z})^2 + 1)^{1/2}} \right]_{z+\Delta z} \tag{3}$$

and

$$\vec{T}_{z+\Delta z} = T \left[\frac{(\frac{\partial y}{\partial z}, 1)}{((\frac{\partial y}{\partial z})^2 + 1)^{1/2}} \right]_{z+\Delta z} \tag{4}$$

At z, the direction of the tension is opposite that at $z + \Delta z$.

$$\vec{T}_z = -T \left[\frac{(\frac{\partial y}{\partial z}, 1)}{((\frac{\partial y}{\partial z})^2 + 1)^{1/2}} \right]_z \tag{5}$$

Now we assume that $\partial y / \partial z$ is much less than 1. This approximation tells us that the pulse must be changing gently or else the answer we get and the process we measure will be different. By limiting the range of validity of our answer, we can replace the denominator of 5 and 4 with 1. We get

$$\Delta \vec{F} = T(\left[\frac{\partial y}{\partial z}\right]_{z+\Delta z} - \left[\frac{\partial y}{\partial z}\right]_{z}, 0) \tag{6}$$

$$\Delta F_y = T(\left[\frac{\partial y}{\partial z}\right]_{z+\Delta z} - \left[\frac{\partial y}{\partial z}\right]_{z}) \tag{7}$$

$$= m\frac{\partial^2 y}{\partial t^2} \tag{8}$$

The mass of the segment of rope is given by $\lambda \Delta l$ which, provided $\partial y/\partial z$ is not too big, is simply $\lambda \Delta z$. Thus we get an equation of motion for y :

$$\lambda \Delta z \frac{\partial^2 y}{\partial t^2} = T(\left[\frac{\partial y}{\partial z}\right]_{z+\Delta z} - \left[\frac{\partial y}{\partial z}\right]_{z}) \tag{9}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\lambda}\frac{(\left[\frac{\partial y}{\partial z}\right]_{z+\Delta z} - \left[\frac{\partial y}{\partial z}\right]_{z})}{\Delta z} \tag{10}$$

$$= \frac{T}{\lambda}\frac{\partial^2 y}{\partial z^2} \tag{11}$$

$$= v^2 \frac{\partial^2 y}{\partial z^2} \tag{12}$$

Here we have made the substitution

$$v = \sqrt{T/\lambda} \tag{13}$$

What is the solution to this equation. Remarkably it is any function of $z - vt$. To see this, we let $u = z - vt$ and $y = f(u)$. Then we have

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) \tag{14}$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}\right) \tag{15}$$

$$= \frac{\partial^2 u}{\partial t^2}\frac{\partial f}{\partial u} + \left(\frac{\partial u}{\partial t}\right)^2 \frac{\partial^2 f}{\partial t^2} \tag{16}$$

$$= v^2 \frac{\partial^2 f}{\partial t^2} \tag{17}$$

$$\frac{\partial^2 y}{\partial z^2} = \frac{\partial^2 u}{\partial z^2}\frac{\partial f}{\partial u} + \left(\frac{\partial u}{\partial z}\right)^2 \frac{\partial^2 f}{\partial z^2} \tag{18}$$

$$= \frac{\partial^2 f}{\partial z^2} \tag{19}$$

This explains why when we pluck a string the wave propogates roughly undistorted down the rope. It also explains why an increase in tension or the use of a thinner string leads to a more rapid propagation.

1.2 The importance of sine and Cosine solutions to the wave equation

The most important solutions to the wave equation is the cosine solution.

$$f(z,t) = A\cos(kz - \omega t + \delta) \tag{20}$$
$$= A\cos(k(z - vt) + \delta) \tag{21}$$

The trivial manipulation of the standard form of 20 to get to 21 shows that $\omega = v/k$. The variable k is of course equal to $2\pi/\lambda$ where λ is the wavelength. The period is defined as the time it takes the cosine function to move a wavelength:

$$v = \omega k \tag{22}$$
$$k = 2\pi/\lambda \tag{23}$$
$$T = \lambda/v \tag{24}$$
$$= 2\pi/\omega \tag{25}$$

It seems silly at first to think of the sine and cosine functions as "more important" as other solutions. There are, however, important physical reasons for considering these functions rather than the general solution. We first note that by considering the cosine solution to the wave equation, we have not limited ourselves. By Fourier's theorem, any function $f(z - vt)$ can be written as a sum of $f(z,t)$ solutions with $\delta = 0$ and $f(z,t)$ solutions with $\delta = -\pi/2$:

$$f(z - vt) = \sum_i A_i \cos(k_i(z - vt)) + B_i \sin(k(z - vt)) \tag{26}$$

(For the case that we are considering the propagation of an unbounded wave, the integral form of Fourier's theorem applies.) The reason the sine and cosine functions become important has to do with the range of validity of the wave equation. As it turns out, in many systems (such as the propagation of light through matter or the disturbance of the displacement of a string) the sine and cosine functions do a great job describing the physics HOWEVER the effective velocity of the wave becomes a function of the frequency. Thus the sine and cosine functions might describe the physics at hand, but an arbitrary function $f(z - vt)$ does not simply propagate as might be predicted.

Thus, when solving a given problem, we can consider simply the sine and cosine solutions, and later sum them to match our initial conditions.

Electromagnetic radiation shares the mathematics of the linear wave equation. For the case of mechanical waves the linear wave equation is usually an approximation that gets us close to the reality of the physical situation on hand. For electromagnetic radiation, as was shown in unit 2, the linear wave equation exactly follows from Maxwells equations and deviations are only seen when the quantum nature of light must be considered.