1 Lecture Notes Unit #2: Maxwell's Equations, Conservation Laws.

1.1 Maxwell's fix

The equations governing electromagnetic fields, as we would write them down in light of what we have learned up to now in this course, read as follows:

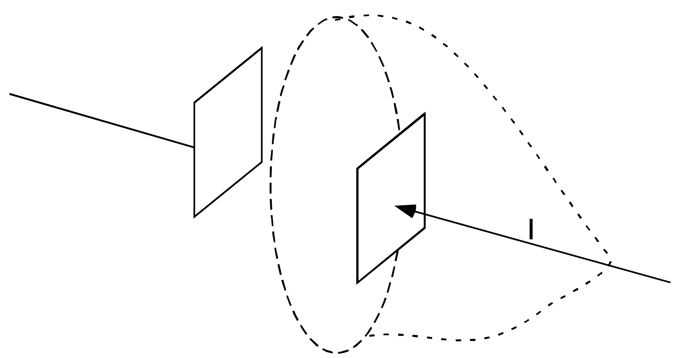
$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} \quad (1)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (2)$$

$$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_o \quad (3)$$

$$\vec{\nabla} \cdot \vec{B} = \vec{0} \quad (4)$$

There is an internal inconsistency in these equations. To see this, consider the simple act of charging up a capacitor:



(5)

This looks like an innocent enough situation until we consider the integral

$$\oint \vec{B} \cdot d\vec{\ell} = \int \vec{\nabla} \times \vec{B} \cdot d\vec{a} \quad (6)$$

$$= \mu_o \int \vec{J} \cdot d\vec{a} \quad (7)$$

$$= \mu_o I_{enc} \quad (8)$$

The second line is definitely true: it simply an application of the mathematical truth of Stokes' law. The second line is true if equation 1 is true. Equation 8 is of course simply Ampere's law. But we have a problem. We know a current can flow into a capacitor (we can observe the build up of charge.) Yet what to we choose for I_{enc}? If we choose the current through a surface that does not cross the wire feeding into the capacitor, then we get $I_{enc} = 0$. A surface cutting through the wire has $I_{enc} = I(t) \neq 0$.

Your first temptation might be to try to fix this problem by determining the "correct" surface, but both enclose the current loop and hence both should work. The problem, incredibly enough, is that Ampere's law does not apply to time-dependent currents.

Let us try to fix Ampere's law for the case of an ideal parallel plate capacitor with $E = V/d$ on the inside and $E = 0$ on the outside. (If we wanted to publish the following result, we would have to be very careful to include the fringe field or prove in the limit of a very large capacitor, its contribution is not important. This is because the ideal capacitor is itself unphysical. This problem is avoided by the approach of problem 7.32 in your text) Let us also make the assumption that the break-down of Ampere's law has something to do with the presence of an electric field. If we use the funky-shaped surface that is far from the plates of the capacitor, $E = 0$ and we might expect that Ampere's law is still true:

$$\mu_o I = \oint \vec{B} \cdot d\vec{\ell} \tag{9}$$

We know the relationship between E inside the capacitor and the change in Voltage with respect to time.

$$\begin{aligned} I &= C\frac{dV}{dt} && (10) \\ E &= \frac{V}{d} && (11) \end{aligned}$$

and

$$C = \frac{A\varepsilon_o}{d} \tag{12}$$

so that

$$\begin{aligned} \mu_o I &= \mu_o \varepsilon_o A \frac{\partial E}{\partial t} && (13) \\ &= \mu_o \varepsilon_o \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} && (14) \\ &= \oint \vec{B} \cdot d\vec{\ell} && (15) \end{aligned}$$

By Stokes' theorem we have found, for the surface that cuts through the center of the capacitor,

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} \tag{16}$$

This is Maxwell's fix to Ampere's law. Maxwell stated that the laws governing electromagnet fields are

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} && (17) \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} && (18) \\ \vec{\nabla} \cdot \vec{E} &= \rho/\varepsilon_o && (19) \\ \vec{\nabla} \cdot \vec{B} &= 0 && (20) \end{aligned}$$

(We should note that this problem was not what motivated Maxwell. He was instead motivated by problems with a now debunked theory of the ether.)

1.2 Implications of Maxwell's Equations

1.2.1 Charge conservation

Maxwell's equations had far reaching implications. One was that now the theory was consistent with charge conservation. To see this note that, if charge is not to be created or destroyed (and this rule has never seen to be broken) the charge change in a volume must be the same as the net charge flowing into that volume:

$$\begin{aligned} \oint \vec{J}\cdot d\vec{a} &= -\frac{dQ}{dt} && (21)\\ &= -\frac{d}{dt}\int \rho dV && (22)\\ &= -\int \frac{\partial \rho}{\partial t} dV && (23) \end{aligned}$$

The negative sign in the last line is simply because the closed integral is defined in terms of the charge flowing out of the volume. Using Gauss' theorem we have

$$\oint \vec{J}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{J}\, dV \tag{24}$$

so that

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{25}$$

Eventually you will recognize equations like eq 25 immediately as a statement of charge conservation. The point is that for electric charge, this rule has NEVER been observed to be broken. Yet before Maxwell's fix, this equation was not consistent with Ampere's law: If we have

$$\vec{\nabla}\times\vec{B} = \mu_o \vec{J} \tag{26}$$

then we must have

$$\begin{aligned} \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) &= \mu_o \vec{\nabla}\cdot\vec{J} && (27)\\ &= 0 && (28)\\ &= \frac{\partial \rho}{\partial t} && (29) \end{aligned}$$

This would imply that $\vec{\nabla}\cdot\vec{J} = 0$ which would in turn imply that the and charge density in space can not change with time without creating or destroying charge. If instead we have

$$\vec{\nabla}\times\vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} \tag{30}$$

then

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_o\vec{\nabla}\cdot\vec{J}+\mu_o\varepsilon_o\frac{\partial\vec{\nabla}\cdot\vec{E}}{\partial t} \quad (31)$$

$$= \mu_o\left(\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}\right) \quad (32)$$

$$= 0 \quad (33)$$

This is great! Maxwell's fix states that it is impossible to create or destroy charge without violating his modified laws of electrodynamics. Although it was not his motivation for coming up with his fix, Maxwell recognized this as an important advantage of his theory.

1.2.2 Electromagnetic radiation in a vacuum

Maxwell also realized that his laws allowed for the existence of electromagnetic radiation in a vacuum. We will get to more of this in unit 3 but, in brief, in a vacuum we have

$$\vec{\nabla}\times\vec{B} = \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t} \quad (34)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad (35)$$

$$\vec{\nabla}\cdot\vec{E} = 0 \quad (36)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad (37)$$

So that

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \quad (38)$$

$$= -\nabla^2\vec{E} \quad (39)$$

$$= -\frac{\partial\vec{\nabla}\times\vec{B}}{\partial t} \quad (40)$$

$$= -\mu_o\varepsilon_o\frac{\partial^2\vec{E}}{\partial t^2}. \quad (41)$$

It is no coincidence that $1/(\varepsilon_o\mu_o)$ is exactly the speed of light squared. Thus Maxwell's equations in a vacuum give us

$$\frac{\partial^2\vec{E}}{\partial t^2} = c^2\nabla^2\vec{E}. \quad (42)$$

This is the three dimensional wave equation! Thus electromagnetic fields can propagate through space and there is such a thing called LIGHT.

1.2.3 Special Relativity

There was one far reaching implication of Maxwell's equations that would have to wait for Einstein. This, as we shall see later in the course, is the realization that Galileo was not quite right in his statement of the invariance of physics under velocity transformations. Instead Maxwell's equations are invariant under another type of transformation (called a Minkowski transformation.) Einstein realized that all of physics INCLUDING mechanics is invariant under the transformation of Minkowski, not Galileo, and came up with a new theory which we will study at the end of the course called the theory of Special relativity.

1.3 Maxwell's Equations and Magnetic Charge

The statement $\vec{\nabla} \cdot \vec{B} = 0$ has the physical interpretation that there is no source of magnetic field in the same way as there is a source of electric field. There is no a-priori reason to assume that this is the case-it's just what we observe. If there was such a thing as magnetic charge so that a particle with this charge would feel a force $q_m \vec{B}$, then Maxwell's equations would be modified as follows:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_e}{\varepsilon_o} \quad (43)$$

$$\vec{\nabla} \cdot \vec{B} = \mu_o \rho_m \quad (44)$$

$$\vec{\nabla} \times \vec{E} = -\mu_o \vec{J}_m - \frac{\partial \vec{B}}{\partial t} \quad (45)$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}_e + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} \quad (46)$$

There are compelling theoretical reasons to believe that monopoles should exist. If you are interested in the hunt for them, talk to Professor Kabfleisch of our department - he's looked for them.

1.4 Maxwell's equations in matter

Maxwell's equations in matter result from an electric polarization per unit volume $\vec{P}$ and magnetic polarization per unit volume $\vec{M}$. You will recall from last semester that these polarizations lead to an effective "bound" charge density

$$\rho_b = -\vec{\nabla} \cdot \vec{P} \quad (47)$$

and current density

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \quad (48)$$

Maxwell's equation for the divergence of the electric field becomes

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_o}(\rho_f + \rho_b) \quad (49)$$

$$= \frac{1}{\varepsilon_o}(\rho_f - \vec{\nabla} \cdot \vec{P}) \quad (50)$$

which is usually written

$$\vec{\nabla} \cdot \vec{D} = \rho_f \tag{51}$$

where

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} \tag{52}$$

is the displacement vector.

The steady-state equation for the curl of $\vec{B}$ is modified in a similar way

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_o(\vec{J}_f + \vec{J}_b) &(53)\\ &= \mu_o(\vec{J}_f + \vec{\nabla} \times \vec{M}) &(54)\end{aligned}$$

which is usually written as

$$\vec{\nabla} \times \vec{H} = \vec{J}_f \tag{55}$$

where

$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M} \tag{56}$$

What do we need to add time dependence to this equation? To get to the time-dependent form of this equation, we must add a new current density $\vec{J}_p$ corresponding the possible change of charge density with respect to time. We expect this current density to obey the continuity equation

$$\begin{aligned} \vec{\nabla} \cdot \vec{J}_p + \frac{\partial \rho_b}{\partial t} &= 0 &(57)\\ &= \vec{\nabla} \cdot (\vec{J}_p - \frac{\partial}{\partial t}\vec{P}) &(58)\end{aligned}$$

As it turns out, the correct choice for $\vec{J}_p$ is simply

$$\vec{J}_p = \frac{\partial}{\partial t}\vec{P} \tag{59}$$

Putting this together, we have

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_o(\vec{J}_f + \vec{J}_b + \vec{J}_p) + \mu_o\varepsilon_o\frac{\partial \vec{E}}{\partial t} &(60)\\ &= \mu_o(\vec{J}_f + \vec{J}_b + \frac{\partial}{\partial t}\vec{P}) + \mu_o\varepsilon_o\frac{\partial((\vec{D} - \vec{P})/\varepsilon_o)}{\partial t} &(61)\\ &= \mu_o(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}) + \mu_o\vec{J}_b &(62)\end{aligned}$$

or, equivalently

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \tag{63}$$

Thus Maxwell's equations in matter are

$$\vec{\nabla} \cdot \vec{D} = \rho_f \tag{64}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{65}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{66}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \tag{67}$$

where

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} \tag{68}$$

$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M} \tag{69}$$

1.5 Conservation of Energy and Poynting's Theorem

Earlier we learned that the energy stored in an electromagnetic field is given by

$$U_{em} = \frac{1}{2}\int \left(\varepsilon_o E^2 + \frac{1}{\mu_o}B^2\right) d\tau \tag{70}$$

(Recall that $d\tau$ is our volume integral.) If you look back in our notes, the electric term is proved in the general case but the magnetic field term is simply derived for a special case and you are told to trust me that it holds in general. HOWEVER, both derivations made use of only the static form of Maxwell's equations and thus this result is true only in the quasi-static approximation, and is exact only if the fields are not changing with time. In short

$$\frac{\partial}{\partial t}U_{em} \neq \frac{\partial}{\partial t}\frac{1}{2}\int \left(\varepsilon_o E^2 + \frac{1}{\mu_o}B^2\right) d\tau \tag{71}$$

Let us find the change in electromagnetic potential energy with respect to time using the time-dependent form of Maxwell's equations. To do this, we first calculate the mechanical work done per unit time on the charges by the electromagnetic field. This is given by

$$\frac{dW}{dt} = \sum_i \vec{F}_i \cdot \vec{v}_i \tag{72}$$

$$= \sum_i q_i(\vec{E}_i + \vec{v}_i \times \vec{B}) \cdot \vec{v}_i \tag{73}$$

$$= \sum_i q_i \vec{E} \cdot \vec{v}_i \tag{74}$$

Now we assume $q_i = \rho d\tau$ and $\vec{J}_i/\rho = \vec{v}_i$ so that

$$\frac{dW}{dt} = \int \vec{E} \cdot \vec{J} d\tau \tag{75}$$

Using

$$\vec{J} = \frac{1}{\mu_o}(\vec{\nabla} \times \vec{B}) - \varepsilon_o \frac{\partial \vec{E}}{\partial t} \tag{76}$$

we have

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_o}\vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \varepsilon_o \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \tag{77}$$

You can easily prove from vector calculus that

$$\vec{E} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \tag{78}$$

Now the fun starts. We can replace $\vec{\nabla} \times \vec{E}$ with $-\frac{\partial \vec{B}}{\partial t}$ and write

$$\vec{E} \cdot \vec{J} = -\frac{1}{\mu_o}\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \varepsilon_o \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} - \frac{1}{\mu_o}\vec{\nabla} \cdot (\vec{E} \times \vec{B}) \tag{79}$$

Finally, using

$$\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t} \tag{80}$$

We have

$$\vec{E} \cdot \vec{J} = -\frac{\varepsilon_o}{2}\frac{\partial E^2}{\partial t} - \frac{1}{2\mu_o}\frac{\partial B^2}{\partial t} - \frac{1}{\mu_o}\vec{\nabla} \cdot (\vec{E} \times \vec{B}) \tag{81}$$

and so the total work done by the electric and magnetic fields are given by

$$\begin{aligned} \frac{dW}{dt} &= -\frac{1}{2}\frac{\partial}{\partial t}\int \left(\varepsilon_o E^2 + \frac{1}{\mu_o}B^2\right) d\tau - \frac{1}{\mu_o}\int \vec{\nabla} \cdot (\vec{E} \times \vec{B}) d\tau & (82) \\ &= -\frac{1}{2}\frac{\partial}{\partial t}\int \left(\varepsilon_o E^2 + \frac{1}{\mu_o}B^2\right) d\tau - \frac{1}{\mu_o}\int (\vec{E} \times \vec{B}) \cdot d\vec{a} & (83) \end{aligned}$$

In the last line we have used the divergence theorem. Now we have calculated the work done by the fields. If the forces due to electromagnetic radiation are to be conservative, then this work comes at a loss of electromagnetic potential energy. We therefore have

$$\begin{aligned} \frac{dU_m}{dt} &= \frac{1}{2}\frac{\partial}{\partial t}\int \left(\varepsilon_o E^2 + \frac{1}{\mu_o}B^2\right) d\tau + \frac{1}{\mu_o}\int (\vec{E} \times \vec{B}) \cdot d\vec{a} & (84) \\ &= \frac{1}{2}\frac{\partial}{\partial t}\int \left(\varepsilon_o E^2 + \frac{1}{\mu_o}B^2\right) d\tau + \int \vec{S} \cdot d\vec{a} & (85) \end{aligned}$$

Here $\vec{S} = \frac{1}{\mu_o}\vec{E} \times \vec{B}$ is the Poynting vector and this equation is the integral form of Poynting's theorem.

The first term is exactly what we would have obtained by differentiating the form for the energy stored in a electromagnetic field in the quasi-static approximation. The second term is the integral of a flux over the area surrounding the volume of interest. It is a flux not of charge, but instead of energy per unit area per unit time. Thus Maxwell's equations allow us to have the transport of energy (say from the sun) without the motion of charge!!!

The work done on charges in a volume will speed them up, slow them down, or somehow change there potential energy. (Maybe they are tethered to springs.) Suppose we account for this change in energy per unit volume per unit time and call it u_{mech}. Now let us define the electromagnetic energy stored per unit volume

$$u_e = \varepsilon_o E^2 + \frac{1}{\mu_o} B^2 \tag{86}$$

This is a very real quantity. It is the energy that is actually stored in the field per unit time- energy that if we are careful, we can get back. At first glance it might seem that $u_e + u_{mech}$ should remain constant as energy sloshes back and forth between mechanical and electrical energy. This is of course, not the case because energy can radiate away. We have

$$\frac{\partial}{\partial t}(u_{mech} + u_e) = -\vec{\nabla} \cdot \vec{S} \tag{87}$$

Equation 87 is the differential form of Poynting's theorem. This is very similar to the continuity equation for charge:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J} \tag{88}$$

Just as the continuity equation is a statement of conservation of charge, Poynting's theorem is a statement of conservation of energy. We have two more factors to work out: Conservation of momentum and conservation of angular momentum.