Physics 4213/5213
Lecture 22

1 Introduction

2 Gauge Invariance of Electromagnetic Fields

Maxwell's equations define both the electric and magnetic fields completely. That is, these equations can be used to calculate the field or these equations can be used to verify that a given field is correct. An alternative method to solving the Maxwell equations directly is to rewrite them in terms of potentials. In many cases, it is easier to solve for the potential first and the determine the fields later.

First, the Maxwell equations are given by:

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{1}$$
$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \qquad \vec{\nabla} \cdot \vec{E} = \rho$$

The two equations on the first line lead to the definition of the potentials

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A} \tag{2}$$
$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \tag{3}$$

Notice, in the case of no time dependent magnetic fields, the relation between the electrostatic potential is recovered. It is important to notice that the fields are given by taking derivatives of the potentials. That is the potential are not uniquely defined, only potential differences matter. The potential $\vec{A}$ can be changed by $\vec{A} \to \vec{A} + \vec{\nabla}\chi$ and the $\vec{B}$ field remains the same. This transformation imposes the condition that Φ

$$\vec{A} \to \vec{A} + \vec{\nabla}\chi \quad \Rightarrow \quad \Phi \to \Phi - \frac{\partial \chi}{\partial t} \tag{4}$$

This is referred to as a gauge transformation.

One consequence of this transformation, is that the photon has to be massless. This is best seen by using the Lorentz covariant formulation of electrodynamics. Recall that the field strength tensor (this is composed of all the components of the electric and magnetic fields) is related to the vector potential as follows:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{5}$$

The Lagrangian that defines Maxwell's equations is given by

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{6}$$

(to prove that this is the correct Lagrangian, use the Euler-Lagrange equation to show that the Maxwell equations are recovered). This equation is both Lorentz invariant and gauge invariant. To add a mass term, recall that in the Klein-Gordon Lagrangian this corresponds the self-interacting field ($\phi^*\phi$). In the case of this equation, the field is the vector potential, so a self-interacting term that is also Lorentz invariant gives a Lagrangian of the form:

$$\mathcal{L} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu \tag{7}$$

The first term in the Lagrangian is gauge invariant, since it leads to Maxwell' equations (also very straight forward to show). On the other hand, the mass term changes under this transformation

$$A^\mu \to A^\mu + \partial^\mu \chi \quad \Rightarrow \quad A^\mu A_\mu \to A^\mu A_\mu + 2A^\mu \partial_\mu \chi + (\partial^\mu \chi)(\partial_\mu \chi) \tag{8}$$

clearly not gauge invariant unless $m = 0$.

3 Gauge Invariance in Quantum Mechanics

Instead of asking what condition does gauge invariance impose on the Klein-Gordon equation (note this is true also of the Schrödinger equation and the Dirac equation), first ask what symmetry the wave-function has. If the wave function is multiplied by a phase, the Lagrangian does not change ($\Phi \to \Phi e^{i\chi}$). This states that the wave-function is invariant to a global phase transformation, that is only phase differences are measurable. But applying this as a global phase transformation doesn't make sense since this says that the phase changes everywhere at the same instant—this is not allowed by relativity.

Instead of applying a global phase transformation, apply a local phase transformation ($\Phi \to \Phi e^{i\chi(x^\mu)}$). The Klein-Gordon equation is not invariant to this transformation due to the derivatives

$$(\partial^\mu \Phi)^\dagger (\partial_\mu \Phi) - m^2 \Phi^* \Phi \to (\partial^\mu \Phi + i\Phi \partial^\mu \chi(x))^\dagger (\partial_\mu \Phi + i\Phi \partial_\mu \chi(x)) - m^2 \Phi^* \Phi \tag{9}$$

The difference is the gradient of a scalar function. If the partial derivative is replaced by $\partial^\mu \to \mathcal{D}^\mu = \partial^\mu + ieA^\mu$ (this is referred to as the gauge covariant derivative) and require transformation $A^\mu \to A^\mu - (1/e)\partial^\mu \chi(x)$ whenever a local phase transformation is performed the Lagrangian is invariant.

This implies that the requirement of local gauge invariance, produces interactions among the scalar fields. The interaction terms can be seen by expanding out the covariant derivative terms:

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger (\partial^\mu \Phi) - m^2 \Phi^* \Phi - ieA^\mu (\Phi^\dagger \partial \Phi - \Phi \partial_\mu \Phi^\dagger) + e^2 A^\mu A_\mu \Phi^\dagger \Phi \tag{10}$$

where the last two terms correspond to the interaction. The first is a scaler current with a single photon, the second is a single scaler current with two photons at the same vertex.

4 Charged Particle in Electromagnetic Field—The Minimal Substitution

Before starting the discussion of gauge invariance, the introduction of the electromagnetic field into the Lagrangian is discussed. First recall that the Lagrangian, in classical mechanics, is defined as the difference between the kinetic and potential energies. The potential energy for a charged particle in a static electric field is given by $q\Phi$. If a magnetic field is present, then the potential has to be derived from the Lorentz force $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$. First substitute the potential for each of the fields:

$$\left.\begin{array}{l}\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A}\end{array}\right\} \Rightarrow \vec{F} = q\left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \vec{\nabla} \times \vec{A}\right) \tag{11}$$

The Lorentz force after some manipulation can be reduced to

$$\vec{F} = q\left\{-\vec{\nabla}\left(\Phi - \vec{v} \cdot \vec{A}\right) - \frac{d}{dt}\left(\vec{\nabla}_v(\vec{v} \cdot \vec{A})\right)\right\} \tag{12}$$

where $\vec{\nabla}_v$ is the gradient in terms of the velocity. This expression in the language of Lagrangian mechanics is a generalized force (that is it does not come from conservative force) it also leads to a Lagrangian of the form

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \Rightarrow L = T - U \quad \text{with} \quad U = q\left(\Phi - \vec{v} \cdot \vec{A}\right) \tag{13}$$

Given the Lagrangian above, the canonical momentum can then be derived. This is given by

$$\frac{\partial L}{\partial \dot{q}_j} = \vec{p} - q\vec{A} \tag{14}$$

Notice that the total energy is given by:

$$E = m + T + q\Phi \tag{15}$$

Since these expressions are totally relativistic, the 4-momenta for a charged particle in an electromagnetic field is obtained by substituting the 4-momenta for a free particle with the canonical momenta

$$p^\mu \rightarrow p^\mu - qA^\mu \tag{16}$$

This expression allows the simple transformation of the Klein-Gordon equation for a free particle to one under the influence of an electromagnetic field. This is referred to as the minimal substitution.