Physics 4213/5213
Lecture 21

1 Introduction

In a previous lecture, a relativistic wave equation was derived. This equation yielded as free particle solution a scalar wave-function. Since it has no degrees of freedom other than the spatial degrees, the equation represents a spin zero particle. Further, the 4-current density had to be reinterpreted from its interpretation in non-relativistic quantum, where it is given a probabilistic interpretation. In the Klein-Gordon equation it is interpreted as a charged current density to avoid the problem of having a negative probability density. Historically, the negative energies and probability density caused the Klein-Gordon equation to be ignored. A second attempt at a relativistic equation was made, by making the equation depend on a first order time derivative, to avoid the negative probability densities. This equation, the Dirac equation will be discussed in this lecture.

2 Dirac Equation

A second attempt at finding an relativistic quantum equation was carried out by Dirac. With the main problem being that the Klein-Gordon equation is second order in the time derivative (recall the original interpretation of the of the current density was in terms of probabilities, and negative probabilities occur), Dirac proposed an equation that was first order in all derivatives. In this way he hoped to eliminate the problem of a negative probability density and maybe also the negative energies.

To start with, he proposed an equation of the following form

$$[\vec{\alpha}\cdot\vec{p}+\beta m]\,\psi = E\psi \Longrightarrow \left[-i\vec{\alpha}\cdot\vec{\nabla}+\beta m\right]\psi = i\frac{\partial\psi}{\partial t}. \tag{1}$$

where the standard quantization conditions are imposed:

$$\vec{p} = -i\vec{\nabla} \qquad E = i\frac{\partial}{\partial t} \tag{2}$$

In order for this to be a valid relativistic equation, it must satisfy the relativistic energy momentum relation for a free particle, that is it must satisfy the Klein-Gordon equation. This will then impose a condition on the $\vec{\alpha}$ and β parameters in equation 1. Squaring equation 1 leads to:

$$\begin{aligned} E^2\psi &= (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)\,\psi \\ E^2\psi &= \left[\alpha_i^2 p_i^2 + (\alpha_i\alpha_j + \alpha_j\alpha_i)\,p_i p_j + (\alpha_i\beta + \beta\alpha_i)\,p_i m + \beta^2 m^2\right]\psi \end{aligned} \tag{3}$$

where the condition $i > j$ is imposed on the second term in equation 3. Comparing equation 3 to the Klein-Gordon equation or the relativistic energy-momentum relation imposes the following condition on the parameters

$$\alpha_i^2 = \beta^2 = 1 \tag{4}$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \tag{5}$$

$$\alpha_i\beta + \beta\alpha_i = 0. \tag{6}$$

Obviously the only way for these relations to hold is for the α_i and β to all be matrices and the wave function cannot be a simple function either.

To determine the form of the matrices, the following conditions need to be imposed:

- The wave function should be a column vector in order that the probability density be easily given as $\psi^\dagger\psi$. This imposes the condition that the matrices must be square.
- The Hamiltonian must be hermitian so that its eigenvalues are real. This forces the α_i and β matrices to also be hermitian: $\alpha_i = \alpha_i^\dagger$ and $\beta = \beta^\dagger$.

Based on these conditions and equation 4 the matrices have eigenvalues ± 1. Further, using the trace theorem $\text{Tr}[AB] = \text{Tr}[BA]$ and the relation $\alpha_i = -\beta\alpha_i\beta$, which comes from equation 5, there are an equal number of eigenvalues with values $+1$ and -1:

$$\text{Tr}[\alpha_i] = \text{Tr}[\beta^2\alpha_i] = \text{Tr}[\beta\alpha_i\beta] = -\text{Tr}[\alpha_i] \Longrightarrow \text{Tr}[\alpha_i] = 0. \tag{7}$$

Based on this result, there are four matrices with an even dimension. Since a dimension of two only gives three anti-commuting matrices, the smallest dimension that fulfills our requirement is four and therefore the wave function must be a column vector of dimension four also. At this point it is not necessary to introduce an explicit representation for the various matrices and column vectors.

2.1 Covariant Form of Dirac Equation

The form of the Dirac equation given above, is not in a form that easily demonstrates its covariance. The main reason being that the time and coordinates are not put on an equal footing. To transform the equation, multiply both sides by β:

$$\left[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right]\psi = i\frac{\partial\psi}{\partial t} \quad \Rightarrow \quad \left[-i\beta\vec{\alpha}\cdot\vec{\nabla} + m\right]\psi = i\beta\frac{\partial\psi}{\partial t} \tag{8}$$

Next introduce the γ matrices $\gamma^\mu = (\beta, \beta\vec{\alpha})$ and rewrite the Dirac equation as:

$$\left[i\gamma^\mu\partial_\mu - m\right]\psi = 0 \tag{9}$$

This equation puts both the time and position coordinates on an equal footing.

Before proceeding, a few properties of the γ matrices are dervied. First the anti-commutation relations are derived. These are derived from the anti-commutation relations of the $\vec{\alpha}$ and β matrices:

$$\begin{array}{ll} \beta\alpha_i + \alpha_i\beta = 0 & (\alpha_i)^2 = (\beta)^2 = 1 \\ \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} & \gamma^\mu = (\beta, \beta\vec{\alpha}) \end{array} \tag{10}$$

Start with the first equation and multiply from the left by β:

$$\beta(\beta\alpha_i) + (\beta\alpha_i)\beta = \gamma^0\gamma^i + \gamma^i\gamma^0 = 0 \tag{11}$$

Now take the second anti-commutation relation in equation 10 and multiply from both the left and right by β:

$$(\beta\alpha_i)(\alpha_j\beta) + (\beta\alpha_j)(\alpha_i\beta) = 2\delta_{ij}\beta\beta \quad \Rightarrow \quad \gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij} \tag{12}$$

where the α β anti-commutation relation was used. Putting the previous two equations together yields:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \tag{13}$$

The hermiticity of the γ^μ matrices can be derived in a manner similar to the commutation relations. Start with γ^0, since it is equal β and β is hermitian $\gamma^{0\dagger} = \gamma^0$—it is hermitian also. The other components are given by:

$$\gamma^{i\dagger} = (\beta\alpha^i)^\dagger = (\alpha^i \beta) = -\gamma^i \tag{14}$$

where the hermiticity of α and β are used, and these components are shown to be anti-hermitian. The hermitian conjugate can therefore be written as:

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 \tag{15}$$

2.2 The Conserved Current

As in the case of the Klein-Gordon equation, a conserved current can be derived. This current is derived in a manner analogous to that in the Klein-Gordon equation. Start with the Dirac equation and take the hermitian conjugate:

$$(i\gamma^\mu \partial_\mu \psi - m\psi = 0)^\dagger \quad \Rightarrow \quad -i\partial_\mu \psi^\dagger \gamma^0 \gamma^\mu \gamma^0 - m\psi^\dagger = 0 \tag{16}$$

Next multiply from the right by γ^0:

$$(-i\partial_\mu \psi^\dagger \gamma^0 \gamma^\mu \gamma^0 - m\psi^\dagger = 0)\gamma^0 \quad \Rightarrow \quad i\partial_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} = 0 \tag{17}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$. To derive the conserved current, multipy the Dirac equation from the left by $\bar{\psi}$ and the hermitian conjugate from the right by ψ. Then then add the two equations together:

$$\left.\begin{array}{l} i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi = 0 \\ i(\partial_\mu \bar{\psi})\gamma^\mu \psi + m\bar{\psi}\psi = 0 \end{array}\right\} \quad \Rightarrow \quad \partial_\mu(\bar{\psi}\gamma^\mu \psi) = \partial_\mu j^\mu = 0 \tag{18}$$

notice, that at this point $\bar{\psi}\gamma^\mu\psi$ has not been shown to be a 4-vector, this will be shown in a later lecture. In principle, $\rho(= j^0)$ in this case is a positive number, of course the normalization has yet to be selected, and so it can again have both values.

2.3 The Gamma Matrices

There are numerous ways of writing the gamma matrices explicitly. Starting with $\beta = \gamma^0$ and the relation that the square is the unit matrix, the eigenvalues of this matrix must be ± 1. This element is taken as diagonal:

$$\beta = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

To satisfy the remaining relations, the alpha matrices are written in terms of the Pauli matrices:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \Rightarrow \gamma^i = \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{20}$$

Note that each element of the γ^μ matrices is a 2×2 matrix. As a reminder, the Pauli matrices are given by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

3 The Free Particle Dirac Wave-Function

Given that the Dirac equation is written in terms of 4×4 matrices and a 4 element column vector, it can be written as 4 coupled differential equations. It has already been shown that each component of the Dirac equation must satisfy the Klein-Gordon equation; this was the condition that was imposed on the Dirac equation to get the form of the γ matrices. Therefore the solution to the Dirac equation must be of the form:

$$\psi(x) = u(p)e^{-ip_\mu x^\mu} \tag{22}$$

where each of the variables are four vectors. This solution is substituted back into the Dirac equation giving:

$$(i\gamma^\mu\partial_\mu - m)u(p)e^{-ip_\mu x^\mu} = (\gamma^\mu p_\mu - m)u(p)e^{-ip_\mu x^\mu} = (\gamma^\mu p_\mu - m)u(p) = 0 \tag{23}$$

being a totally algebraic equation.

To arrive at the form of the free particle solutions, the matrix representation of the Dirac equation is used:

$$\begin{aligned}\gamma^\mu p_\mu &= E\begin{pmatrix}\mathbf{I} & 0\\ 0 & -\mathbf{I}\end{pmatrix} - \vec{p}\cdot\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix} = \begin{pmatrix}E\mathbf{I} & -\vec{p}\cdot\vec{\sigma}\\ \vec{p}\cdot\vec{\sigma} & -E\mathbf{I}\end{pmatrix}\\ \Rightarrow (\gamma^\mu p_\mu - m)\,u(p) &= \begin{pmatrix}E-m & -\vec{p}\cdot\vec{\sigma}\\ \vec{p}\cdot\vec{\sigma} & -E-m\end{pmatrix}\begin{pmatrix}u_A\\ u_B\end{pmatrix} = \begin{pmatrix}(E-m)u_A - \vec{p}\cdot\vec{\sigma}u_B\\ \vec{p}\cdot\vec{\sigma}u_A - (E+m)u_B\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\end{aligned} \tag{24}$$

notice that the equation is broken into two pieces, even though this is a four component equation. Breaking up the two pieces gives:

$$u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E-m}u_B, \qquad u_B = \frac{\vec{p}\cdot\vec{\sigma}}{E+m}u_A \tag{25}$$

which finally leads to:

$$u_A = \frac{(\vec{p}\cdot\vec{\sigma})^2}{E^2-m^2}u_A \tag{26}$$

This does not give a value for the wave-function, but it does impose a condition on the energy and momentum. This condition can be found by expanding out the numerator:

$$\vec{p}\cdot\vec{\sigma} = p_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + p_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + p_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}p_z & p_x - ip_y\\ p_x + ip_y & -p_z\end{pmatrix} \tag{27}$$

$$\Rightarrow (\vec{p}\cdot\vec{\sigma})^2 = \mathbf{I}|\vec{p}|^2 \Rightarrow u_A = \frac{|\vec{p}|^2}{E^2-m^2}\mathbf{I}u_A \tag{28}$$

This equation imposes the condition that $E = \pm\sqrt{|\vec{p}|^2 + m^2}$, which is what would be expected. The positive energy, as before, is associated with the particle state while the negative energy is associated with the anti-particle state.

Finally the wave-function is determined. Notice that there is some level of arbitrariness in the solution. The only requirement is that they be orthogonal and that they have energy eigenvalues corresponding to the positive and negative energy solutions. Going back to the relation between u_A and u_B and imposing that the solutions be orthogonal gives:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_B = \frac{1}{E+m}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \qquad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u_B = \frac{1}{E+m}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$
$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_A = \frac{1}{E-m}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \qquad u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u_A = \frac{1}{E-m}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} \tag{29}$$

Notice that the first two equations must have $E > 0$ otherwise the solution blows up when $\vec{p} = 0$. For the second two equations the energy must be less than zero ($E < 0$) otherwise the solution blows up when $\vec{p} = 0$. The particle solutions are therefore:

$$u^1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}, \qquad u^2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \tag{30}$$

while the anti-particle solutions are:

$$u^3 = N \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}, \qquad u^4 = N \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} \tag{31}$$

The prescription that has been used so far is to redefine the negative energy solutions as positive energy anti-particles. New states are then defined as:

$$v^1 = u^4(-p) = N \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \qquad v^2 = -u^3(-p) = -N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \tag{32}$$

Notice that as the momentum approaches zero, the momentum dependent term approaches zero, which then gives the non-relativistic solution used in the Schrödinger equation. Finally the equations that govern the particle and anti-particle solutions are given as:

$$(\gamma^\mu p_\mu - m)\, u(p) = 0, \qquad (\gamma^\mu p_\mu + m)\, v(p) = 0 \tag{33}$$

The wave-function normalization has yet to be selected. The normalization will be chosen the same as for bosons:

$$\int \rho dV = \int \psi^\dagger \psi dV = u^\dagger u = 2E \tag{34}$$

where the number of particles per unit volume is given by the expression above. Applying this condition to the spinors gives $N = \sqrt{|E| + m}$.