Physics 4213/5213
Lecture 20

1 Introduction

In the previous lectures, a theory and method of electromagnetic interactions was developed. The theory has been primarily applied to the case of particle scattering. In this lecture a simple theory of weak interactions is developed. This theory will then be applied to the case of particle decays, in particular the case of pion and kaon decays.

2 A Simple Theory of Weak Interactions

The first observed example of a weak decay, was what is popularly referred to as nuclear β-decay. The reaction involved the nucleus of the atom changing by one unit of charge, and the emission of an electron (positron) plus a neutrino; initially the neutrino was not observed, but was postulated in order that energy and momentum be conserved. The reaction as viewed initially, was postulated to represent the decay of either a neutron or a proton through the following modes:

$$\begin{aligned} n &\rightarrow p + e^- + \bar{\nu}_e \\ p &\rightarrow n + e^+ + \nu_e \end{aligned} \tag{1}$$

note that the second reaction can only occur for a proton bound in a nucleus.

These reactions were all characterized by a single coupling constant (G_F). Further, other processes such as neutrino scattering are also characterized by the same coupling strength. In neutrino scattering, the cross section was also found to be proportional to the center of mass energy squared:

$$\sigma_{\nu+p\rightarrow n+e^+} \propto G_F^2 s \tag{2}$$

This is unlike the case of electron-electron scattering that is proportional to the inverse of an energy squared. This implies that G_F is inversely proportional to a fixed mass (α_W/M^2). Therefore, to make the weak interaction analogous to the electromagnetic interaction, the exchanged particle is characterized by $1/q^2$ where $q^2 = M^2$ a fixed value, as opposed to the variable virtual photon mass.

Based on a desire to make a theory that describes the data, and one that is similar in structure to the electromagnetic interaction, the Feynman rules for this theory require the following change from the electromagnetic case: e^2/q^2 by G_F.

3 Composite Particle Decay and Decay Constants

The first question that must be answered, is how to treat the decay of a composite particle. In the case to be treated here $\pi^- \rightarrow \ell^- + \bar{\nu}_\ell$, the π^- is known to be composed of a quark and an antiquark ($d\bar{u}$). These in principle annihilate producing a virtual W that then decays to the lepton and neutrino. The problem with doing the calculation in this manner, is that the forces that bind the two quarks is ignored, but the this force is large compared to all other forces in the problem, so must be taken into account in some manner.

To see how the pion with all its binding energy is accounted for, recall that this term becomes part of the invariant amplitude. The invariant amplitude is a Lorentz scalar, that is it is composed of the product of two 4-vectors. The pieces of the invariant amplitude that are known are the factor accounting for the weak interaction, and the pieces associated with the lepton vertex:

$$\mathcal{M} = G_F (P_\ell - P_\nu)^\mu \tag{3}$$

To complete this another 4-vector is needed. This 4-vector must describe the pion in some manner. The only 4-vector that is available, is its 4-momentum q_μ. This can still be multiplied by a scalar function, which can at most be a function of its 4-momentum squared (mass). Again this is the only kinematic variable available to describe the pion. Therefore the pion is described by the form-factor:

$$f_\pi(m_\pi) q_\mu = f_\pi q_\mu \tag{4}$$

where the last equality is due to the mass of the pion being a constant.

4 The Decay of the Pion

Having come up with a method of describing the pion, the calculation of the decay width can proceed. Starting with the form-factor given above the invariant amplitude is:

$$\mathcal{M}_{fi} = G_F f_\pi q_\mu (P_\ell - P_\nu)^\mu = G_F f_\pi m_\pi (E_\ell - E_\nu) \tag{5}$$

which is clearly a scalar.

Given the amplitude, the decay width is given with factors similar to the cross section. This includes a density of states factor for each outgoing particle, and a normalization associated with the total number of initial particles. The density of states factors are the same as used in the cross section, but the initial flux is replaced by the total number of initial particles $2m_\pi$. This makes the differential decay width:

$$d\Gamma = \frac{|\mathcal{M}_{fi}|^2}{2m_\pi} (2\pi)^4 \delta(P_\pi - P_\ell - P_\nu) \frac{d^3p_\ell}{(2\pi)^3 2E_\ell} \frac{d^3p_\nu}{(2\pi)^3 2E_\nu} \tag{6}$$

Since the total width is desired, the integrals over both momenta have to be carried out. Integrating over the lepton momentum imposes momentum conservation on the system; three of the delta-functions are removed. This leaves:

$$d\Gamma = \frac{|\mathcal{M}_{fi}|^2}{2m_\pi} (2\pi)^4 \delta(E_\pi - E_\ell - E_\nu) \frac{1}{(2\pi)^3 2E_\ell} \frac{d^3p_\nu}{(2\pi)^3 2E_\nu} \tag{7}$$

Since the delta function contains only energies, the integral must be converted to an integral over energy. The following change of variables is used:

$$|\mathbf{p}_\nu|^2 = E_\nu^2 - m_\nu^2 \quad \Rightarrow \quad |\mathbf{p}_\nu| dp_\nu = E_\nu dE_\nu \tag{8}$$

This leads to:

$$d\Gamma = \frac{|\mathcal{M}_{fi}|^2}{2m_\pi} (2\pi)^4 \delta(E_\pi - E_\ell - E_\nu) \frac{1}{(2\pi)^3 2E_\ell} \frac{E_\nu dE_\nu d\Omega}{(2\pi)^3 2} \tag{9}$$

where the fact that the neutrino is massless was used.

The final integral can now be carried out. This leaves:

$$\Gamma = \frac{|\mathcal{M}_{fi}|^2}{8\pi m_\pi} \frac{E_\nu}{E_\ell} \tag{10}$$

4.1 2-Body Decay Kinematics

The expression for the decay width is now given in terms of the energy of the two leptons. Since this is a two body decay mode, the energies must be constants that are related to the masses of the three particles. To determine tha relation, start with energy-momentum conservation laws:

$$m_\pi = E_\nu + E_\ell \qquad P_\ell = P_\nu \tag{11}$$

The second relations can be written in terms of the energy:

$$P_\ell = P_\nu \quad \Rightarrow \quad E_\ell^2 - m_\ell^2 = E_\nu^2 \tag{12}$$

Solving the two equations for the energies yields:

$$E_\ell = \frac{m_\pi^2 + m_\ell^2}{2m_\pi} \qquad E_\nu = \frac{m_\pi^2 - m_\ell^2}{2m_\pi} \tag{13}$$

4.2 The Decay Rate

Taking equation 10 for the the decay width, and the relation for the two energies gives the total decay rate equal to:

$$\Gamma = \frac{G_F^2 f_\pi^2 m_\pi}{8\pi} \left(\frac{m_\ell^2}{m_\pi}\right)^2 \left(\frac{m_\pi^2 - m_\ell^2}{m_\pi^2 + m_\ell^2}\right) \tag{14}$$