Physics 4213/5213
Lecture 19 (Part 2)

1 Introduction

In the last lecture, the invariant amplitude for both e^-e^- and e^-e^+ scattering were derived. Each of the processes were found to require two Feynman diagrams in order to describe them. The e^-e^- mode required the two diagrams since the two outgoing electrons are indistinguishable, and therefore which e^- comes from a given vertex cannot be determined. This required placing each of the outgoing electrons at the two possible vertices. The e^+e^- process requires two Feynman diagrams, since the e^+e^- can annihilate each other, or they can scattering off each other. Both these cases are indistinguishable in that the same initial particles are used, and the same outgoing particles are seen.

After drawing the Feynman diagrams, the invariant amplitudes were written down using the Feynman rules previously given. The amplitudes were found to be related to each other, through the relation:

$$\mathcal{M}_{e^-e^+}(P_A, P_B, P_C, P_D) = \mathcal{M}_{e^-e^-}(P_A, -P_D, P_C, -P_B) \tag{1}$$

It is this relation that will be examined more closely in this lecture. It will be examined through the Mandelstam variables.

2 The Mandelstam Variables

In high energy physics, cross sections and decay rates are written using kinematic variables that are relativistic invariants. In the process at hand, two body elastic scattering, there are in fact four such invariants available. These are the 4-momenta associated with each particle, but since energy and momentum are conserved, only two of these are necessary to define the kinematics. Typically, the variables used are combinations of the four, these are defined as follows:

$$\begin{aligned} s &= (P_A + P_B)^2 \\ t &= (P_A - P_C)^2 \\ u &= (P_A - P_D)^2 \end{aligned} \tag{2}$$

where the reaction $A + B \rightarrow C + D$ defines the particles involved in the reaction (see also fig 1). These quantities are not all independent of each other (only two as previously stated). The sum of the three variables are equal to the sum of the masses:

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 \tag{3}$$

Now consider the problem of e^+e^- scattering. In this case all the masses are identical. Further, the three variables are identified with the following physical variables:

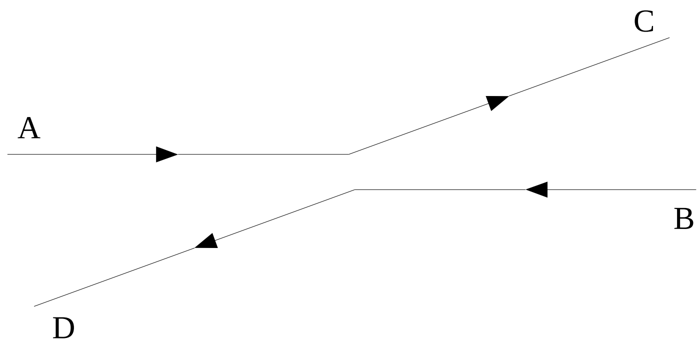

Figure 1: Definition of particles involved in the scattering process.

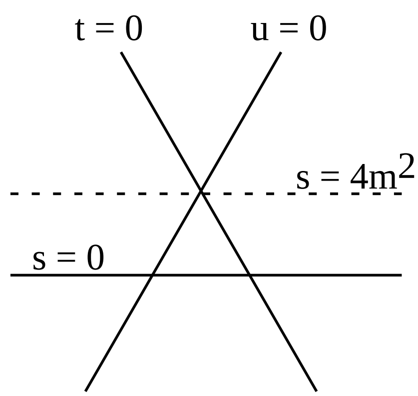

Figure 2: The figure shows the valid regions of phase space allowed by energy and momentum conservation.

Variable	Definition	Value in CM Frame
s	the center of mass energy	$s = (E_A + E_B)^2$
t	the 4-momentum transfer of e^+ or e^-	$-2p^2(1 - \cos\theta)$
u	the 4-momentum difference between e^+ and e^-	$-2p^2(1 + \cos\theta)$

Further, to find the physically allowed values for the variables, start by realizing that both t and u are negative and have maximum values of 0. For this case the variable $s = 4m_e$, which corresponds to the case where the outgoing particles are produced at rest. For values of $s > 0$, the variables are bound to the region between $t = 0$ and $u = 0$ (see fig 2).

Now notice that figure 2 also shows two additional triangular regions, but with $t_{\min} = 4m^2$ and $u_{\min} = 4m^2$. To determine what these regions correspond to physically, make the following transformation for the region with $u_{\min} \geq 4m_e^2$, $P_D \to -P_D$, which makes:

$$u = (P_A + P_D)^2 \tag{4}$$

and make the transformation $P_B \to -P_B$ so that s can go down to zero, making it:

$$s = (P_A - P_B)^2 \tag{5}$$

and t remains the same. Effectively what has happened, is that the reaction $A + B \to C + D$ has been transformed to $A + \bar{D} \to C + \bar{B}$, since the negative of the 4-momenta is the 4-momenta of the antiparticle. This is referred to as the u-channel reaction.

The remaining region gives the t-channel reaction, where $P_B \to -P_B$ and $P_C \to -P_C$. In this case the reaction is transformed from $A + B \to C + D$ to $A + \bar{C} \to \bar{B} + D$. Therefore, each one of the regions starts with the original particles, moves them from one side of the reaction to the other, which converts particles into antiparticles. This cycles the three variables, making each positive in a different region, and each correspond to the center of mass energy in that region. As a concrete example, consider the case where $A = e^-$, $B = e^+$, $C = e^-$, and $D = e^+$. The s, t, and u channels are then defined as:

Channel	Reaction
s	$e^- + e^+ \to e^- + e^+$
t	$e^- + e^+ \to e^+ + e^-$
u	$e^- + e^- \to e^- + e^-$

This makes the connection between e^+e^-, and e^-e^- scattering obvious.

3 The e^+e^-, e^- e^- Scattering Amplitude

Having arrived at the connection between the three Mandelstam variables in terms of particle and antiparticle reactions, it would seem obvious that the invariant amplitudes should be written in terms of the three Mandelstam variables. Take the s-channel Feynman diagram to be process under study for each of the amplitudes. Then the three variables are defined as in equation 2. With these definitions, the invariant amplitudes become:

$$\begin{aligned} \mathcal{M}_{e^+e^-} &= e^2\left(\frac{s-u}{t} + \frac{t-u}{s}\right) \\ \mathcal{M}_{e^-e^-} &= e^2\left(\frac{u-s}{t} + \frac{t-s}{u}\right) \end{aligned} \tag{6}$$

For each process, the mass of the photon is written as one of the Mandelstam variables. For e^+e^- scattering, the annihilation term has the photon mass written as s, while the for scattering diagram the virtual photon mass is given by t.

For the e^-e^- process, virtual photon masses are given in terms of t, and u. The t channel photon diagram, corresponds primarily to forward scattering due to the $1/(1-\cos\theta)$ term, while the u channel photon diagram contributes primarily to backward scattering due to the $1/(1+\cos\theta)$ term.