Physics 4213/5213
Lecture 18

1 Introduction

In the last few lectures a method for calculating differential and total cross sections has been developed. The method was developed for the case of the electromagnetic interaction, but in fact is more general, and can be applied to almost any interaction as long as the coupling constant is small. The method is based several approximation:

1. That the beam is prepared a large distance from the scattering center, so that a plane wave solution can be used;
2. That the outgoing particles are measured a large distance from the scattering center, so that a plane wave solution can be used;
3. The time variations of the potential are large compared to the interaction time;
4. That only terms to first order in the charge are kept in the transition amplitude.

Based on these approximations, the general form of the differential cross section is given by:

$$d\sigma = \frac{|\mathcal{M}|^2}{F} dQ \tag{1}$$

where the flux factor F is given by:

$$F = |\mathbf{v}_A - \mathbf{v}_B| 2E_A 2E_B \tag{2}$$

and the phase space factor dQ is given by:

$$dQ = (2\pi)^4 \delta^4 (P_B + P_D - P_A - P_C) \frac{d^3 p_C}{(2\pi)^3 2E_C} \frac{d^3 p_D}{(2\pi)^3 2E_D} \tag{3}$$

where this describes two outgoing particles (for the general case, there is one density of states factor per outgoing particle, and the delta function describes energy momentum conservation).

In this lecture, the Rutherford cross section will be derived. This cross section describes the scattering of a very light particle (small compared to the energy) off a very heavy particle that is initially at rest. The derivation will not assume this but we will put it in as a last step, so that the general form of the cross section can be examined.

2 The Rutherford Cross Section

The calculation starts with writing down the Feynman diagram that describes the process (see fig. 1). First write down the invariant amplitude (matrix element):

$$-i\mathcal{M} = [ie(P_A + P_B)^\mu] \left(-i \frac{g_{\mu\nu}}{q^2} \right) [ie(P_C + P_D)^\nu] \tag{4}$$

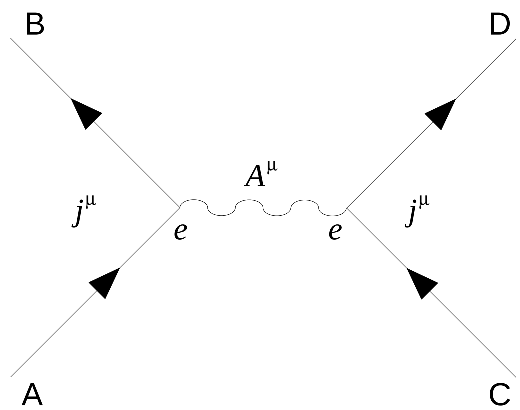

Figure 1: This is the Feynman diagram that is being calculated in this lecture note.

where:

$$q^2 = (P_B - P_A)^\mu (P_B - P_A)_\mu = (P_C - P_D)^\mu (P_C - P_D)_\mu \tag{5}$$

Next comes the flux factor, it is given by:

$$F = |v_A| 2E_A 2E_C = 2|\mathbf{p}_A| 2m_C \qquad \text{where} \qquad |\mathbf{v}| = \frac{|\mathbf{p}_A|}{E_A} \tag{6}$$

Finally the phase space factor is given by:

$$dQ = (2\pi)^4 \delta^4 (P_B + P_D - P_A - P_C) \frac{d^3 p_B}{(2\pi)^3 2E_B} \frac{d^3 p_D}{(2\pi)^3 2E_D} \tag{7}$$

With all the factors written out explicitly, the differential cross section can be calculated. But before proceeding, it must be decided what quantity is to be measured. In this problem, particle A is incident on particle C, which is at rest. After the scatter particle A becomes particle B, while particle C becomes particle D; note that A and B are the same particle, the label is used to distinguish before and after, with the same holding for particles C and D. Since particle D is assumed to be very heavy, it will not travel far so will most likely be unmeasurable, therefore the only quantity that will be measurable will be particle B. The cross section will then be given in terms of the angular distribution of particle B relative to the incident particle A.

Given the quantity that is to be measured, the first step is to do the integral over the 3-momenta of particle D. This integral imposes momentum conservation, through the delta-function; the condition imposed is $\mathbf{p}_D = \mathbf{p}_A - \mathbf{p}_B + \mathbf{p}_C$. At this point in the calculation, the differential cross section is in the form:

$$d\sigma = \frac{\delta(E_B + E_D - E_A - E_C) |\mathcal{M}_{if}|^2}{64\pi^2 m_C^2 |\mathbf{p}_A| E_B} |\mathbf{p}_B|^2 dp_B d\Omega \tag{8}$$

where the relation $d^3 p_B = |\mathbf{p}_B|^2 dp_B d\Omega$ has been used. Since the delta-function involves the energy, and there is an energy (E_B) in the equation (note that the energy depends on the momentum), the integral should be converted from a momentum integral to one over the energy. The change of variables can be achieved through the relation:

$$|\mathbf{p}_B|^2 = E_B^2 - m_B^2 \quad \Rightarrow \quad 2|\mathbf{p}_B|dp_B = 2E_B dE_B \tag{9}$$

With this change of variables, the delta function can be integrated out. This yields:

$$d\sigma = \frac{|\mathcal{M}_{fi}|^2}{64\pi^2 m_C^2}\frac{|\mathbf{p}_B|}{|\mathbf{p}_A|}d\Omega \tag{10}$$

notice that the invariant amplitude is just carried along, since the it contains only 4-momenta, and the integrals over the delta function impose energy-momentum conservation.

The final step is to simplify the matrix element. First of all simplify q^2. Since the final result is to be written in terms of the kinematic variables of particle A-B, q^2 is given by:

$$\begin{aligned} q^2 =& (P_B - P_A)^\mu (P_B - P_A)_\mu = 2m^2 - 2E_i E_f + 2|\mathbf{p}_i||\mathbf{p}_f|\cos\theta \\ \Rightarrow& \quad 4E_i E_f \sin^2\theta/2 \qquad \text{for } E \gg m \end{aligned} \tag{11}$$

note at this point the subscripts are changed to indicate that particles A and B are the same, and define the initial and final kinematic quantities. The numerator, which is the sum of 4-momenta, as given in equation 4, is simplified by using energy-momentum conservation to remove particle D the final state of particle C:

$$\begin{aligned} e^2(P_A + P_B)^\mu (P_C + P_D)_\mu &= e^2(P_A + P_B)^\mu (2P_C + P_A - P_B)_\mu \\ &= e^2(2P_A \cdot P_C + m^2 - P_A \cdot P_B + 2P_B \cdot P_C + P_B \cdot P_A - m^2) \\ &= 2e^2(E_i + E_f)m_C \end{aligned} \tag{12}$$

Finally, from these expressions, the differential cross section is given by:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}\frac{(E_i + E_f)^2}{(m^2 - E_i E_f + |\mathbf{p}_i||\mathbf{p}_f|\cos\theta)^2} \tag{13}$$

where the relation $\alpha = e^2/4\pi$ was used. In the limit where the energy of particle A is much larger than its mass, and the mass of the target particle is extremely large, the differential cross section is given by:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4}\frac{1}{E^2 \sin^4\theta/2} \tag{14}$$

where the kinematics yields the relations $E_i = E_f$, and $|\mathbf{p}_i| = |\mathbf{p}_f|$. This equation is referred to as the Rutherford cross section.