Physics 4213/5213
Lecture 17 (Part 2)

1 Introduction

In the previous few lectures, the tools needed to describe a scattering process of relativistic charged particles has been developed. In this lecture all the pieces are put together to give the general formula for the cross section of two charged particles scattering through the interaction of an electromagnetic field. In addition, the Feynman rules for the scattering of spinless charged particles through the electromagnetic interaction will be given. The Feynman rules that are derived, are general and can be used to describe any scattering process and decay process involving photons and spinless particles.

2 Spinless Electron Scattering

Given that the Klein-Gordon equation has solutions that are similar to those of the Schrödinger equation, the transition amplitude T_{fi} for relativistic particles is the same as for the non-relativistic case. The transition amplitude for the non-relativistic case was found to be:

$$T_{fi} = -i \int_{-T/2}^{T/2} e^{-i(E_i - E_f)t} dt \int_V \phi_f^*(\vec{r}) V(\vec{r}, t) \phi_i(\vec{r}) d^3x \tag{1}$$

What needs to be done for the relativistic case, is to find an appropriate potential to substitute in this equation. Notice that it is being assumed that this equation is valid in the relativistic case.

First rewrite the Klein-Gordon equation in covariant form:

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi = 0 \tag{2}$$

where $\partial^\mu = \left(\partial/\partial t, -\vec{\nabla}\right)$ and $\partial_\mu = \left(\partial/\partial t, \vec{\nabla}\right)$ and like up down indices are summed over. Since the interest is in calculating electromagnetic interactions, the electromagnetic field must be introduced into the problem. This is done by using the minimal substitution derived in the previous lecture $mu^\mu \to p^\mu + eA^\mu$. Next apply the quantum condition, by converting the canonical momentum to an operator as ($\partial^\mu \to \partial^\mu - ieA^\mu$), and substituting into the Klein-Gordon equation:

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi = -V\phi, \qquad V = -ie\left(\partial_\mu A^\mu + A^\mu \partial_\mu\right) - e^2 A^2 \tag{3}$$

where the charge of the electron ($q = -e$) has been selected. Note also, that the sign of V was selected so that the non-relativistic approximation of the Klein-Gordon equation gives the Schrödingerequation.

The potential given in equation 3 is characterized by the charge e; notice that in the early discussions of the Feynman diagrams, the amplitude was characterized by e^2 meaning that there is sill an addition charge to be found. As a first approximation to finding the transition amplitude, the term with the e^2 is ignored. The potential above is substituted into the formula for the transition amplitude where it has been rewritten in covariant form:

$$T_{fi} = i \int ie\phi_f^*(\vec{r}, t) \left(\partial_\mu A^\mu + A^\mu \partial_\mu\right) \phi_i(\vec{r}, t) d^4x \tag{4}$$

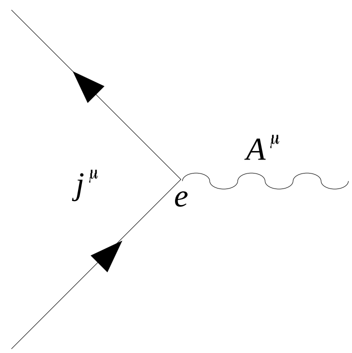

Figure 1: This gives a graphical representation of equation 7.

Integrating the first term by parts gives:

$$T_{fi} = i \int ieA^\mu \left[\phi_f^*(\vec{r},t)\partial_\mu\phi_i(\vec{r},t) - \phi_i(\vec{r},t)\partial_\mu\phi_f^*(\vec{r},t)\right] d^4x \tag{5}$$

This can be simplified by noting that the probability density and current can be written in covariant form as:

$$\begin{aligned} &\rho = iq\left(\phi^*\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\phi^*}{\partial t^2}\right), \qquad \vec{j} = -iq\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right) \\ &\Rightarrow j^\mu = iq\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) \end{aligned} \tag{6}$$

Therefore the transition amplitude can be written in a simplified form as:

$$T_{fi} = -i \int A^\mu j_\mu^{fi} d^4x \tag{7}$$

For a free particle the current density can be written out in a simple form. Start with the wave functions for the incoming and outgoing particles:

$$\phi_i = N_i e^{ip_i^\mu x_\mu}, \qquad \phi_f = N_f e^{ip_f^\mu x_\mu} \tag{8}$$

Substituting these into the expression for the current density gives:

$$j^\mu = -eN_iN_f\left(p_i + p_f\right)_\mu e^{i\left(p_f - p_i\right)_\mu x^\mu} \tag{9}$$

Finally this can be looked at pictorially as a current coming in with a fixed momentum interacting with the field and leaving with a different momentum.

3 The Electromagnetic Field

Before the transition amplitude can be calculated, the form of A^μ needs to be determined. The quickest way to finding what its general form should be, is to draw the Feynman diagram for this reaction. As shown below, it corresponds to two currents interacting through the exchange of a

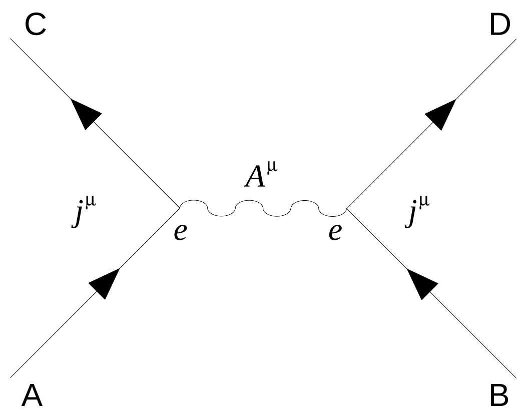

Figure 2: Feynman diagram depicting a two to two process

single photon. Therefore A^μ should be proportional to the current of one of the two particles times some factor describing the photon. The factor describing the photon should be a function of the 4-momentum transferred between the two particles.

To arrive at the analytic form of A^μ start with the following equation:

$$\partial^\mu \partial_\mu A^\nu = j_2^\nu \tag{10}$$

where j_2^ν is the current that generates the field A^ν. The form of the current has already been given in equation 9 as:

$$j_2^\mu = -eN_B N_D (p_B + p_D)^\mu e^{i(p_D - p_B)^\nu x_\nu} \tag{11}$$

where the charge of the electron is used ($q = -e$). For the equation defining the potential to be valid, it must have the form:

$$A^\mu = -\frac{1}{q^2} j_2^\mu, \qquad q^2 = (p_D - p_B)^\mu (p_D - p_B)_\mu \tag{12}$$

which is as expected. So finally the transition amplitude is given by:

$$T_{fi} = -i \int j_\mu^1 \left(-\frac{1}{q^2} \right) j_2^\mu d^4x = -iN_A N_B N_C N_D (2\pi)^4 \delta^4 (p_D + p_C - p_A - p_B) \mathcal{M}_{fi} \tag{13}$$

where the delta function comes from the integration and $\mathcal{M}_{fi}$ the invariant amplitude is defined as:

$$-i\mathcal{M}_{fi} = (ie(p_A + p_C)^\mu) \left(-i \frac{g_{\mu\nu}}{q^2} \right) (ie(p_B + p_D)^\nu) \tag{14}$$

3.1 The Feynman Rules

The invariant amplitude derived above can be used to arrive at rules for calculating Feynman diagrams. These rules can then be used whenever a similar type of line or vertex occurs. Each external line corresponding to a spin zero particle will have a 1 associated with it; in reality an N_i, but the normalization will cancel out. Each vertex between a spin zero particle and a photon will have a factor $ie(p_A + p_B)^\mu$, while each internal photon will have a factor $-ig^{\mu\nu}/q^2$; make sure to include the correct momenta, in each of the factors.

4 The Transition Rate

Before the cross section can be calculated, the normalization factors N need to be fixed. First of all, recall that the probability density of particles described by the wave function is given by $\rho = 2(-e)E|N|^2$. Further, recall that the factor E was needed to compensate for the Lorentz contraction of the volume d^3x to keep the number of particles, ρd^3x, constant. Therefore, the normalization is set such that in a volume V there are $2E$ particles (this is makes sense, since dividing through by the charge leaves number of particles), this implies that $N = 1/\sqrt{V}$.

Keeping in mind that the cross section is related to the square of the transition amplitude and that the square of the transition amplitude involves the square of a delta function which is infinite, the quantity of interest is the transition amplitude per unit volume per unit time; in this case the delta function is four dimensional and requires dividing by the time and volume to be finite. The transition amplitude per unit volume per unit time is given by:

$$W_{fi} = \frac{|T_{fi}|^2}{TV} = (2\pi)^4 \frac{\delta^4(p_C + p_D - p_A - p_B)}{V^4} |\mathcal{M}_{fi}|^2 \tag{15}$$

5 The Cross Section

Finally, the cross section can be calculated. It is related to the transition rate per unit volume through the density of final states and the incident flux:

$$\text{cross section} = \frac{W_{fi}}{\text{inital flux}} (\text{number of final states}) \tag{16}$$

For a single particle, the number of final states in a volume V and momenta in element d^3p is given by:

$$\text{Number of final states} = \frac{V d^3p}{(2\pi)^3} \tag{17}$$

where periodic boundary conditions are used—the allowed states in one dimension are given by $Lp_x = 2\pi n$ with n an integer and if confined to a momentum range $p_x \to p_x + dp_x$ the allowed number of states is $Ldp_x/(2\pi)$ or in three dimensions $Vd^3p/(2\pi)^3$. Since there are $2E$ particles, the number of final states per particle is:

$$\text{Number of final states/particle} = \frac{V d^3p}{2E(2\pi)^3} \tag{18}$$

Finally, given that there are two distinct particles in the final state, the density of final states is given by:

$$\text{Number of avalible final states} = \frac{V d^3p}{2E_C(2\pi)^3} \frac{V d^3p}{2E_D(2\pi)^3} \tag{19}$$

Next the initial flux is calculated. This includes the number of particles incident on a given target. This is most easily calculated in the frame where the target is at rest and the beam has a velocity $\vec{v}_A$. The number of particles crossing a unit area per unit time is given by $2E_A|\vec{v}_A|/V$. The number of target particles per unit volume is $2E_B/V$. Therefore the initial flux is:

$$\text{Initial flux} = |\vec{v}_A|\frac{2E_A}{V}\frac{2E_B}{V} \tag{20}$$

Combining the transition rate per unit volume with the density of initial and final states gives the differential cross section:

$$\begin{aligned} d\sigma &= \frac{V^2}{2E_A 2E_B|\vec{v}_A|}\frac{(2\pi)^4}{V^4}\delta^4(p_C + p_D - p_A - p_B)|\mathcal{M}_{fi}|^2\frac{V^2}{(2\pi)^6}\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D} \\ \Rightarrow d\sigma &= \frac{(2\pi)^4}{2E_A 2E_B|\vec{v}_A|}\delta^4(p_C + p_D - p_A - p_B)|\mathcal{M}_{fi}|^2\frac{d^3p_C}{(2\pi)^3 2E_C}\frac{d^3p_D}{(2\pi)^3 2E_D} \end{aligned} \tag{21}$$

Symbolically the cross section can be written as:

$$d\sigma = \frac{|\mathcal{M}_{fi}|^2}{F}dQ \tag{22}$$

Where the factors are identified as, the Lorentz invariant phase space factor:

$$dQ = (2\pi)^4\delta^4(p_C + p_D - p_A - p_B)\frac{d^3p_C}{(2\pi)^3 2E_C}\frac{d^3p_D}{(2\pi)^3 2E_D} \tag{23}$$

the incident flux (in the laboratory frame):

$$F = |\vec{v}_A| 2E_A 2E_B \tag{24}$$

and the invariant amplitude $|\mathcal{M}_{fi}|^2$ which contains the dynamics of the problem.