Physics 4213/5213
Lecture 17 (Part 1)

1 Introduction

This lecture shows how to add the electromagnetic field into the Klein-Gordon equation. In the case of non-relativistic quantum mechanics, the potential is easily inserted into the Schrödingerequation, it is just the Coulomb potential. In the case of relativistic quantum mechanics, not just the electric potential has to be accounted for, but the magnetic fields generated by the currents has to be taken into account. This will be done through the use of Lagrangian mechanics.

In addition, this lecture will introduce the conditions imposed on the Klein-Gordon equation through the imposition that it remain invariant under local phase transformations. This condition will be shown to be connected to the imposition of the equations being invariant to gauge transformations.

2 Charged Particle in Electromagnetic Field—The Minimal Substitution

Before starting the discussion of gauge invariance, the introduction of the electromagnetic field into the Klein-Gordon equation is discussed. The simplest starting point is the Lorentz force:

$$\left.\begin{array}{l}\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\end{array}\right\} \Rightarrow \mathbf{F} = q\left(-\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times \boldsymbol{\nabla} \times \mathbf{A}\right) \tag{1}$$

The triple cross product is simplified using the vector relation:

$$\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{C}) = \boldsymbol{\nabla}(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{C} - \mathbf{C} \times (\boldsymbol{\nabla} \times \mathbf{B}) - (\mathbf{C} \cdot \boldsymbol{\nabla})\mathbf{B} \tag{2}$$

which gives:

$$\Rightarrow \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} \tag{3}$$

where the fact that the velocity does not depend explicitly on the position is used. Next the total time derivative of the $\mathbf{A}$ is calculated:

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} \tag{4}$$

where the first terms comes from the explicit time dependents of $\mathbf{A}$, while the second term stems from the value of the potential at the position of the particle comes from the motion of the particle. Putting these equations together leads to:

$$\mathbf{F} = e\left[-\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) - \frac{d\mathbf{A}}{dt}\right] \tag{5}$$

which represents a generalized force due to a velocity dependent potential.

To determine the potential, the Lagrangian formalism is introduced. First recall that the Lagrangian is the difference in kinetic and potential energies. Then recall that to get the generalized force, it is the gradient of the potential that is sought:

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \Rightarrow L = T - U \tag{6}$$

To put the generalized force into a more useful form, the total time derivative of the vector potential is rewritten as:

$$\frac{d\mathbf{A}}{dt} = \frac{d}{dt}\boldsymbol{\nabla}_v(\mathbf{A}\cdot\mathbf{v}) \tag{7}$$

where $\boldsymbol{\nabla}_v$ is the gradient in terms of the velocity, and the fact that the vector potential is independent of the velocity is used. The force is now written as:

$$\mathbf{F} = q\left\{-\boldsymbol{\nabla}\left(\Phi - \mathbf{v}\cdot\mathbf{A}\right) - \frac{d}{dt}\left(\boldsymbol{\nabla}_v(\mathbf{v}\cdot\mathbf{A})\right)\right\} \tag{8}$$

and since the scalar potential is independent of the velocity, the generalized potential is identified as:

$$U = q(\Phi - \mathbf{v}\cdot\mathbf{A}) \tag{9}$$

At this point everything that is necessary to include the electromagnetic potentials into the Klein-Gordon equation is available. The final step comes in recognizing that the relation $\mathbf{p} \to -i\boldsymbol{\nabla}$ is the quantum condition on the canonical momentum and not the mechanical momentum (mv). This comes from the Lagrangian, and is given by:

$$\frac{\partial L}{\partial \dot{q}_j} = m\mathbf{v} + q\mathbf{A} \tag{10}$$

In the original derivation of the Klein-Gordon equation, the mechanical energy is used; even though in that case it is equal to the canonical energy. Therefore, to include the electromagnetic potentials, and maintain the quantum condition, the mechanical momentum is replaced with the canonical momentum and the vector potential:

$$m\mathbf{v} \to \mathbf{p} - q\mathbf{A} \quad \Rightarrow \quad mu^\mu \to p^\mu + qA^\mu \quad \Rightarrow \quad -i\partial^\mu \to -i(\partial^\mu - iq\mathbf{A})^\mu \tag{11}$$

where u^μ is the 4-velocity.

3 Gauge Invariance of Electromagnetic Fields

Maxwell's equations define both the electric and magnetic fields completely. That is, these equations can be used to calculate the field or these equations can be used to verify that a given field is correct. An alternative method to solving the Maxwell equations directly is to rewrite them in terms of potentials. In many cases, it is easier to solve for the potential first and then determine the fields later.

The Maxwell equations are given by:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j} & \nabla \cdot \mathbf{E} &= \rho \end{aligned} \tag{12}$$

The two equations on the first line lead to the definition of the potentials

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{13}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{14}$$

Notice, in the case of no time dependent magnetic fields, the relation between the electrostatic potential is recovered. It is important to notice that the fields are given by taking derivatives of the potentials. That is the potential are not uniquely defined, only potential differences matter. The potential $\mathbf{A}$ can be changed by $\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$ and the $\mathbf{B}$ field remains the same. This transformation imposes the condition that Φ transform as follows:

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi \quad \Rightarrow \quad \Phi \rightarrow \Phi - \frac{\partial \chi}{\partial t} \quad \Rightarrow \quad A^\mu \rightarrow A^\mu - \partial^\mu \chi \tag{15}$$

This is referred to as a gauge transformation.

One consequence of this transformation, is that the photon has to be massless. This is best seen by using the Lorentz covariant formulation of electrodynamics. Recall that the field strength tensor (this is composed of all the components of the electric and magnetic fields) is related to the vector potential as follows:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{16}$$

The Lagrangian that defines Maxwell's equations is given by

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{17}$$

(to prove that this is the correct Lagrangian, use the Euler-Lagrange equation to show that the Maxwell equations are recovered). This equation is both Lorentz invariant and gauge invariant. To add a mass term, recall that in the Klein-Gordon Lagrangian this corresponds to the self-interacting field ($\Phi^*\Phi$). In the case of this equation, the field is the vector potential, so a self-interacting term that is also Lorentz invariant gives a Lagrangian of the form:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu \tag{18}$$

The first term in the Lagrangian is gauge invariant, since it leads to Maxwell' equations (also very straightforward to show). On the other hand, the mass term changes under this transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \quad \Rightarrow \quad A^\mu A_\mu \rightarrow A^\mu A_\mu + 2A^\mu \partial_\mu \chi + (\partial^\mu \chi)(\partial_\mu \chi) \tag{19}$$

clearly not gauge invariant unless $m = 0$.

4 Gauge Invariance in Quantum Mechanics

Now that the procedure for incorporating the fields into the Klein-Gordon equation is known, and the transformation properties of the potential under which the fields remain invariant, the properties of the Klein-Gordon can be studied. These properties are more easily studied using the Lagrangian density, since it is the symmetries of the Lagrangian that lead to the various conservation laws. The Lagrangian density is given by:

$$\mathcal{L} = (\partial^\mu \Phi)^*(\partial_\mu \Phi) - m^2 \Phi^* \Phi \tag{20}$$

From Lagrangian mechanics, and symmetry (transformation that does not change the Lagrangian) leads to a conservation law. A simply symmetry of the Lagrangian above is to transform the wave-function by a phase:

$$\Phi \to e^{i\alpha}\Phi \tag{21}$$

This may seem like it should have no consequence, but in fact this transformation leads to a conserved charge. This would imply that this phase transformation must be connected to electromagnetism, except that this transformation does not make physical sense since it implies that the phase of the wave-function changes everywhere in space at the same time. This is forbidden by special relativity.

A slightly more complicated transformation, and one that will satisfy relativity, is that the phase depend on position and time:

$$\Phi \to e^{i\alpha(x^\mu)}\Phi \tag{22}$$

Applying this transformation to the Lagrangian, shows that the Lagrangian is not invariant to a local phase transformation:

$$\mathcal{L} = (\partial^\mu \Phi + i\Phi\partial^\mu \chi(x))^*(\partial_\mu \Phi + i\Phi\partial_\mu \chi(x)) - m^2 \Phi^* \Phi \tag{23}$$

Yet a local phase transformation makes more sense than a global phase transformation under relativity. The question then becomes, what is required for the Lagrangian to remain invariant under a local phase transformation.

First of all notice that the extra piece is the derivative of a scalar function. This is just the piece that is added to the potentials under a gauge transformation. If the electromagnetic field is added to the Lagrangian, the derivatives are transformed as in equation 11. Further, imposing the condition that the potential undergo a gauge transformation while the wave-function undergoes a local phase transformation, leaves the Lagrangian in the same form. Therefore, instead of using the regular derivative in the Klein-Gordon equation, the covariant derivative is used $\mathcal{D}^\mu = \partial^\mu - ieA^\mu$. Notice that the requirement that the Lagrangian be invariant to a local phase transformation, imposed the requirement that an interaction be introduced.

Finally the Klein-Gordon Lagrangian is given by:

$$\mathcal{L} = (\partial^\mu \Phi)^*(\partial^\mu \Phi) - m^2 \Phi^* \Phi - ieA^\mu(\Phi^*\partial\Phi - \Phi\partial_\mu \Phi^*) + e^2 A^\mu A_\mu \Phi^* \Phi \tag{24}$$

where the last two terms correspond to the interaction. The first is a scaler current with a single photon, the second is a single scaler current with two photons at the same vertex.