Physics 4213/5213
Lecture 16

1 Introduction

In the last lecture, the transition probability per unit time was calculated non-relativistically. The derivation made several approximation. These approximations were all connected with how experiments are actually performed. That is the initial state is prepared far from the point where the interaction occurs and the measurement of the final state is also measured far from the interaction. Further it was assumed that the time over which the interaction occurs is small compared to other times in the problem. The last point that needs to be emphasized is that this was done non-relativistically, therefore the transition is from a given initial state to a given finally state for a single particle. Further notice that whatever created the potential remains the same and is not allowed to change, at least the way the transition rate was calculated.

The step to making the "Golden Rule" relativistic and therefore allow the creation of new particle is fairly straight forward. All the mathematical steps and approximations are similar to those used in the non-relativistic case. The only new feature, will be the introduction of the relativistic Klein-Gordon equation, which describes the behavior of spin zero particles.

2 The Klein-Gordon Equation

In analogy with the Schrödinger equation, the relativistic wave equation is given by equating the Hamiltonian to the total energy and replacing the energy and momentum by the appropriate operators:

$$\vec{p} \to -i\vec{\nabla} \qquad\qquad E \to i\frac{\partial}{\partial t} \tag{1}$$

The free particle Klein-Gordon equation is therefore:

$$E^2 = |\vec{p}|^2 + m^2 \quad \Rightarrow \quad \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = m^2\phi \quad \Rightarrow \quad \left(\partial_\mu\partial^\mu + m^2\right)\phi = 0 \tag{2}$$

The solutions to this equation are basically the same as for the Schrödinger equation, except that the energy can now have both positive and negative values. The positive and negative energy values come from the energy and momentum being related through the square of their values:

$$E^2 = |\vec{p}|^2 + m^2 \quad \to \quad E = \pm\sqrt{|\vec{p}|^2 + m^2} \tag{3}$$

With these conditions, the solution to the Klein-Gordon equation is:

$$\phi = Ne^{i\vec{p}\cdot\vec{x} - Et} = Ne^{-ip^\mu x_\mu} \tag{4}$$

where E can be both positive and negative, and N is the normalization factor; in the non-relativistic case it is usually ignored but in the relativistic case this is associated with the number of particles the wave function describes. It should be noted that like the solution to the Schrödingerequation, this equation represents a spin zero particle.

Next the probability density is derived. As a reminder, for the non-relativistic case the probability density and probability current density are:

$$\rho = |N|^2, \qquad \vec{j} = \frac{\vec{p}}{m}|N|^2 \tag{5}$$

For a free particle $N = 1/\sqrt{V}$ in order to insure the probability of finding the particle in a given volume is one. For the relativistic case, multiply both sides of the Klein-Gordon equation by $-i\phi^*$ and the complex conjugate equation by $-i\phi$ and then take the difference:

$$\left.\begin{array}{l} -i\phi^*\nabla^2\phi + i\phi^*\frac{\partial^2\phi}{\partial t^2} = -im^2\phi^*\phi \\ -i\phi\nabla^2\phi^* + i\phi\frac{\partial^2\phi^*}{\partial t^2} = -im^2\phi\phi^* \end{array}\right\} \Rightarrow \frac{\partial}{\partial t}\left[i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi}{\partial t}\right)\right]$$
$$+\vec{\nabla}\cdot\left[-i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)\right] = 0 \tag{6}$$

The final result has the form of the continuity equation, where the terms are identified with the probability density and the probability current density as:

$$\rho = i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right), \qquad \vec{j} = -i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right) \tag{7}$$

Finally the probability density and the probability current density for a free relativistic wave function are:

$$\rho = 2E|N|^2, \qquad \vec{j} = 2\vec{p}|N|^2 \tag{8}$$

Notice the extra factor of E in the probability density (for now ignore the fact the energy can be negative, this is actually taken into account in the interpretation of backward going particles). The extra factor of E insures that the probability is the same in any Lorentz frame. To see this, take a box that is moving at some velocity β in a direction along one of its edges. In this case $d^3x \to d^3x/\gamma$, but $\rho \to \rho\gamma$ since $\gamma \propto E$, then:

$$\text{Rest Frame:} \qquad \int_V 2m|N|^2 d^3x$$
$$\text{Boosted Frame:} \qquad \int_V 2E|N|^2(1/\gamma)d^3x = \int_V 2E|N|^2\left(\frac{m}{E}\right)d^3x = \int_V 2m|N|^2 d^3x \tag{9}$$

3 Negative Energy Solutions

As has already been pointed out, the Klein-Gordon has negative energy solutions. These solutions cannot be ignored, since a complete set of functions is needed to describe a system (recall that any function can be expanded in a complete set). Further, if the negative energy solutions are kept, the probability density will be negative.

To get around this problem, the negative energy states are interpreted as particle traveling backward in time or as positive energy anti-particles traveling forward in time. This doesn't change the solutions:

$$e^{-iEt}, \qquad E > 0 \Rightarrow e^{-i(-E)(-t)}. \tag{10}$$

To get around the problem of a negative probability density, multiply it and the current density by the charge. If the energy is negative, then associate the charge with the sign of the energy. A negative energy state gives a particle of opposite charge:

$$\rho = -e\left[2E|N|^2\right], \quad E > 0 \rightarrow \rho = +e\left[2|E||N|^2\right], \quad E < 0. \tag{11}$$