Physics 4213/5213
Lecture 15

1 Introduction

In the previous lectures, the scattering amplitude was derived in terms of two parameters η_ℓ and δ_ℓ. These parameters describe how an arbitrary potential affects the wave-function, but no mechanism for deriving these parameters was presented. Without definite predictions for these parameters, the most that will come out of this model is a classification of the different scattering processes in terms of the two parameters. This of course is useful information that can be used to build theories, but does not allow tests of current theories.

In this and the next few lectures, a method of calculating cross sections will be developed. This method will be connected to the Feynman diagrams that have already been discussed. The theory will be initially developed for relativistic spinless particles interacting with the electromagnetic field. To start the derivation, Fermi's golden rule is derived. Even though it is derived in the context of the Schrödingerequation, it also holds in the relativistic case.

2 The Golden Rule

For a particle scattering off a potential, the Schrodinger equation is given by:

$$[H_0 + V(\vec{r}, t)]\,\psi(\vec{r}, t) = i\frac{\partial\psi(\vec{r}, t)}{\partial t} \tag{1}$$

where ψ is the wave function. Depending on how complicated the potential is, this may or may not be easy to calculate. Yet in scattering experiments, the measured incident and final state particles are a very large distance from the scattering center. Based on this assumption the problem can be turned into calculating the transition probability of an initially free particle state to a final free particle state. Given this assumption, the wave function ψ can be expanded out in terms of the free particle wave function, which is given by:

$$H_0\phi_n = E_n\phi_n \tag{2}$$

where H_0 is the free particle Hamiltonian. Remember that the ϕ_n are orthogonal to each other:

$$\langle\phi_n|\phi_m\rangle = \delta_{nm} \tag{3}$$

Therefore, unless an interaction term is added to the Hamiltonian, there can be no transition from one free particle state to another.

The wave function for the complete Hamiltonian in terms of the free particle wave functions is:

$$\psi(\vec{r}, t) = \sum_j a_j(t)\phi_j(\vec{r})e^{-iE_jt} \tag{4}$$

Where the initial state of the system is given by:

$$\psi(\vec{r}, -T/2) = \sum_j a_j(-T/2)\phi_j(\vec{r}) = \phi_n(\vec{r}) \tag{5}$$

To calculate the coefficients for an arbitrary time, substitute equation 4 into the Schrodinger equation (eq. 1):

$$\sum_j a_j(t)\left(E_j + V(\vec{r},t)\right)\phi_j(\vec{r})e^{-iE_jt} = i\sum_j\left[(-iE_j)a_j(t) + \frac{da_j(t)}{dt}\right]\phi_j(\vec{r})e^{-iE_jt}$$

$$\Rightarrow \sum_j a_j(t)V(\vec{r},t)\phi_j(\vec{r})e^{-iE_jt} = i\sum_j \frac{da_j(t)}{dt}\phi_j(\vec{r})e^{-iE_jt} \tag{6}$$

Next multiply both sides by ϕ_f^* and integrate over all space (note, only that portion of space where the potential is non-zero contributes to the integral):

$$\frac{da_f(t)}{dt} = -i\sum_j a_j(t)e^{-i(E_j-E_f)t}\int_V \phi_f^*(\vec{r})V(\vec{r},t)\phi_j(\vec{r})d^3x \tag{7}$$

where the integral connects any two states; for now this corresponds to an arbitrary time.

This equation is extremely difficult to solve since it has two unknowns. One way around this, is to find an approximation to this equation that has only one coefficient. Once this is done this solution can be inserted back into the equation, and solving the new equation continuing the iteration process up to the desired level of accuracy. First consider the potential to be finite in range, with an effective range that is much smaller than distance over which the experiment is to be performed. Next assume that the potential is independent of time, or at least that the time variation is small compared to the interaction time. (For the strong interaction, the range is 10^{-13} cm, while the interaction time is typically 10^{-23} sec). Therefore consider the following initial conditions for the wave-function:

$$\psi(\vec{r},-T/2) = \sum_j a_j(-T/2)\phi_j(\vec{r}) = \phi_i(\vec{r}) \tag{8}$$

Imposing this condition on the derivative of $a_f(t)$ gives:

$$\frac{da_f(t)}{dt} = -ie^{-i(E_i-E_f)t}\int_V \phi_f^*(\vec{r})V(\vec{r})\phi_i(\vec{r})d^3x \tag{9}$$

This last equation can be integrated out over time, to give $a_f(t)$:

$$a_f(T/2) = -i\int_{-T/2}^{T/2} e^{-i(E_i-E_f)t}dt\int_V \phi_f^*(\vec{r})V\phi_i(\vec{r})d^3x \tag{10}$$

Finally, since the potential is being assumed independent of time, the time integral on the right hand side can be evaluated:

$$T_{fi} \equiv a_f(T/2) = 2M_{fi}\left[\frac{\sin\left[(E_f-E_i)T/2\right]}{(E_f-E_i)}\right] \tag{11}$$

At this point, it would make sense to define $|T_{fi}|^2$ as the transition probability. But a problem occurs with the energy term in equation 11, due to the approximation being made. This approximation assumes that the initial and final states occur in the limit $T \to \infty$. In this limit, the energy

term becomes $2\pi\delta(E_f - E_i)$, making $|T_{fi}|^2$ a meaningless quantity due to the square of the delta function.

One way around this is to define the transition probability per unit time:

$$W = \lim_{T\to\infty} \frac{|T_{fi}|^2}{T} \tag{12}$$

In this case, the two time integrations, one for each of the T_{fi} in the magnitude squared, are carried out separately:

$$\begin{aligned}\lim_{T\to\infty} \frac{|T_{fi}|^2}{T} &= |M_{fi}|^2 2\pi\delta(E_f - E_i) \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{-i(E_f - E_i)t} dt \\ &= |M_{fi}|^2 2\pi\delta(E_f - E_i) \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt = 2\pi |M_{fi}|^2 \delta(E_f - E_i)\end{aligned} \tag{13}$$

where the first equation on the second line is due to the delta function being zero everywhere except at $E_i = E_f$. Even though this equation has a delta function, its purpose is to impose energy conservation, this equation is far better behaved than that for $|T_{fi}|^2$.

To finish this off, and take care of the delta function, W is multiplied by the density of possible final states $\rho(E_f)$. Remember, that after the particle is scattered, it will go into some final state. The probability of a specific state depends on the number of final states available and how closely spaced they are. The transition rate is therefore given by:

$$W_{fi} = 2\pi \int |M_{fi}|^2 \delta(E_f - E_i) \rho(E_f) dE_f \tag{14}$$

which is referred to as **Fermi's Golden Rule**.

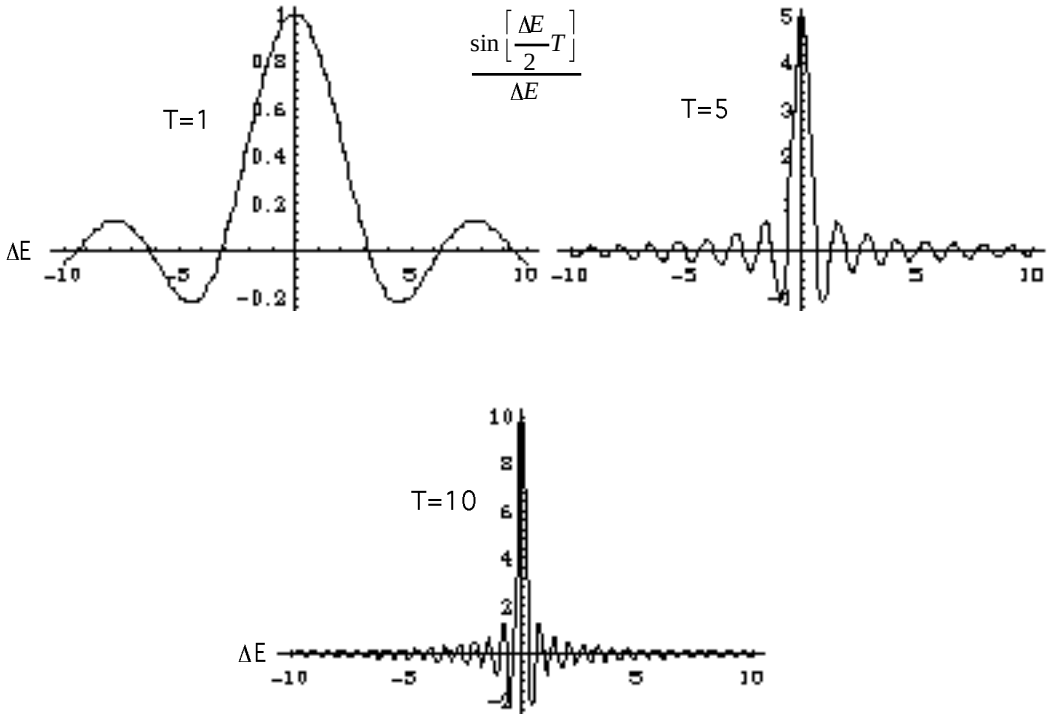

Figure 1: The evolution of the $\sin[(E_i - E_f)T]/(E_i - E_f)$ as $T \to \infty$. This is the definition of the Dirac delta function.