Physics 4213/5213
Lecture 14 (Part 1)

1 Introduction

The next few lectures will discuss cross sections in a more quantitative way than has been done until now. The discussion will start with a wave function in terms of radial states, which allows the introduction of angular momentum into the problem. This expansion will be used to describe the incident and scattered particles from a scattering center. Note that this treatment will not be relativistic, but it will introduce many of the concepts of scattering cross section. In addition, the idea of resonance production and why they are identified as particles will also be examined.

2 Statement of the Problem

The problem that is to be examined, is the scattering of particles off of other particles. The initial state particles are in a prepared state, that is, the energy, momentum and type of particles are known. On the other hand, the properties of outgoing particles are not known a priori. Another important point, is that the interaction occurs in a region of $\approx 10^{-13}$ cm while the initial state particles are prepared at a minimum a few cm away and the final state system is measured at a minimum also a few cm away.

3 The Incident Beam

The incident beam of particles is describe by a plane wave wave-function:

$$\psi_i = e^{ikz} = e^{ikr\cos\theta} \tag{1}$$

To both introduce angular momentum and to write the solution in an asymptotic form (form for large distances compared to the scattering center), the wave-function is expanded out in terms of the Legendre polynomials;

$$e^{ikr\cos\theta} = \sum_{\ell} a_\ell(kr) P_\ell(\cos\theta) \tag{2}$$

Calculating the coefficients is achieved by multiplying both sides by $P_\ell(\cos\theta)$ and integrating over $d\cos\theta$ from -1 to $+1$. This yields (without proof):

$$a_\ell(kr) = \int_{-1}^{+1} e^{ikr\cos\theta} P_\ell(\cos\theta) = i^\ell (2\ell+1) j_\ell(kr) \tag{3}$$

This leads to

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta) \tag{4}$$

Given equation 4, the condition that the observation of the wave is at a large distance ($kr >> 1$) is imposed, giving;

$$\psi_{\text{inc}} = \frac{i}{2kr} \sum_{\ell} (2\ell + 1) \left[(-1)^{\ell} e^{-ikr} - e^{ikr} \right] P_{\ell}(\cos\theta) \tag{5}$$

Therefore the plane wave is represented by both an incoming and outgoing wave—recall that the plane wave represents a wave covering all space. The first term in the brackets represents the incoming wave, while the second term represents the outgoing wave. Now consider a scattering potential at $r = 0$. In this case, the amplitude and phase of the outgoing wave can be altered. The wave-function then becomes

$$\psi_{\text{total}} = \frac{i}{2kr} \sum_{\ell} (2\ell + 1) \left[(-1)^{\ell} e^{-ikr} - \eta_{\ell} e^{2i\delta_{\ell}} e^{ikr} \right] P_{\ell}(\cos\theta) \tag{6}$$

where $0 \leq \eta_{\ell} \leq 1$—$\eta_{\ell}$ not equal to one represents the case where part of the beam is absorbed. Since this represents the total wave-function the scattered part is obtained by subtracting out the incident wave;

$$\psi_{\text{scat}} = \frac{e^{ikr}}{r} \sum_{\ell} (2\ell + 1) \frac{\eta_{\ell} e^{2i\delta_{\ell}} - 1}{2ki} P_{\ell}(\cos\theta) = \frac{e^{ikr}}{r} F(\theta) \tag{7}$$

where $F(\theta)$ is referred as the scattering amplitude.

Next, the scattering amplitude needs to be expressed as a cross section. To do so, the flux first needs to be determined. This is just the current that crosses some finite area. The current for the Schrodinger equation is defined as:

$$\vec{j} = \frac{1}{2im} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) \tag{8}$$

For the incident wave this is given by:

$$\vec{j} = \frac{\vec{k}}{m} \tag{9}$$

which corresponds to the flux per unit area across a surface (it is assumed that the current is normal to the surface, this does not change any of the results other than to add a factor of $\cos\theta$ where θ is the angle between the beam and the normal to the surface) while the current for the outgoing wave is:

$$\vec{j} = \frac{\vec{k}}{m} \frac{|F(\theta)|^2}{r^2} \tag{10}$$

The differential cross section is the outgoing flux through a given differential area:

$$\mathcal{F} = \vec{j} \cdot d\vec{A} = \frac{|\vec{k}|}{m} \frac{|F(\theta)|^2}{r^2} \left(r^2 d\Omega \right) \tag{11}$$

divided by the incoming flux per unit area ($\vec{j}$):

$$d\sigma = \frac{|F(\theta)|^2}{r^2} r^2 d\Omega \qquad \Rightarrow \qquad \frac{d\sigma}{d\Omega} = |F(\theta)|^2 \tag{12}$$

Notice that the magnitude of the momentum k has been given the same value both before and after the collision. This implies the scattering is elastic, either off a very heavy particle or in the center of mass frame. But none the less, this cross section corresponds to elastic scattering.

The total elastic cross section can be derived by integrating out equation 12. This is done by realizing that the Legendre polynomials are orthogonal to each other:

$$\int_0^{2\pi}\int_{-1}^{1} P_\ell(\cos\theta)P_{\ell'}(\cos\theta)d\phi d\cos\theta = \frac{4\pi\delta_{\ell\ell'}}{2\ell+1} \tag{13}$$

Finally, this gives:

$$\sigma_{\text{elastic}} = \frac{\pi}{k^2}\sum_\ell(2\ell+1)(1+\eta_\ell^2-2\eta_\ell\cos 2\delta_\ell(k)) \tag{14}$$

If $\eta_\ell = 1$ then all the particles are scattered elastically and there is no absorption:

$$\sigma_{\text{elastic}} = \frac{4\pi}{k^2}\sum_\ell(2\ell+1)sin^2\delta_\ell \tag{15}$$

Further, for the case of no absorption a couple of points need to be made, if $\delta_\ell = 0$ then that particular partial wave does not contribute to the scattering process. The second point, is that the maximum value for any partial wave occurs when $\delta_\ell = \pi/2$.

4 The Total Cross section

To determine the total cross section, the inelastic part has to be added to the elastic part of the cross section. The inelastic part can be derived by looking at the inward radial flux in equation 6 (ψ_{scat}) and comparing it to the outward radial flux. The inward radial flux is given by:

$$\mathcal{F}_{\text{in}} = \int \vec{j}\cdot r^2 d\Omega = \frac{k}{m}\frac{4\pi}{4k^2}\sum_\ell(2\ell+1) \tag{16}$$

while the outward flux is given by:

$$\mathcal{F}_{\text{out}} = \int \vec{j}\cdot r^2 d\Omega = \frac{k}{m}\frac{4\eta_\ell^2\pi}{4k^2}\sum_\ell(2\ell+1) \tag{17}$$

The net loss in flux is the difference between the inward and outward flux and the total inelastic cross section is the difference divided by the incident flux:

$$\sigma_{\text{inel}} = \frac{\pi}{k^2}\sum_\ell(2\ell+1)(1-\eta_\ell^2) \tag{18}$$

Given the elastic and inelastic cross sections, the total cross section is:

$$\sigma_{\text{total}} = \sigma_{\text{el}} + \sigma_{\text{inel}} = \frac{2\pi}{k^2}\sum_\ell(2\ell+1)(1-\eta_\ell\cos 2\delta_\ell) \tag{19}$$

Comparing this equation to the scattering amplitude (note that this is related to the elastic cross section), it is found that

$$\sigma_{\text{total}} = \frac{4\pi}{k}\text{Im}F(0) \tag{20}$$

This is referred to as the optical theorem.

4.1 Maximum Cross Sections

From the expressions given above, it is obvious that there are maximum values that the cross sections can achieve. For the case of no absorption, $\eta_\ell = 1$, the scattering is completely elastic. In this case the maximum value of the total cross section is

$$\sigma_{\text{total}} = \sigma_{\text{el}} = \frac{4\pi}{k^2}\sum_\ell (2\ell + 1) \tag{21}$$

Notice that for the cases of electron-electron and electron-positron elastic scattering, the cross sections based on dimensional analysis are $\sigma \propto \alpha_{\text{em}}^2/s$, which is consistent with equation 21. Also, the scattering is dominated by single photon exchange, corresponding to $\ell = 1$. For the scattering of ν off leptons or quarks, the total cross section depends linearly on the energy (lab frame) $\sigma \propto E$. This expression will eventually exceed the limit given in equation 22.

For the case where the absorption is a maximum $\eta_\ell = 0$, the maximum value of the cross section is

$$\sigma_{\text{total}} = \frac{2\pi}{k^2}\sum_\ell (2\ell + 1) \tag{22}$$

which is simply equal to twice the elastic cross section, therefore the elastic and inelastic cross sections are equal.

5 Experimental Determination of the Total Cross Section

To determine the total cross section, all that is needed is to determine the imaginary part of the forward scattering amplitude. Several problems exist with this, first it is almost impossible to measure what part of the particles going forward are scattered an which are beam going straight through. The second problem is that cross sections are measured and not amplitudes, so there is no simple method to disentangle the real from the imaginary parts of the scattering amplitude.

To actually perform such a measurement, the total rate for particles colliding can be determined. Since this is the imaginary part of the amplitude, the real part can be determined by extrapolating the elastic differential cross section to $\theta = 0$. But measuring the total rate is very difficult experimentally. The method that is used, is a combination of total rate measurements and measurement of the elastic differential cross section. In the case of pp and p$\bar{p}$ (in fact this is true for all hadron-hadron scattering), the differential cross section has the functional form of

$$\frac{d\sigma}{dt} = Ae^{-bt} \tag{23}$$

where $t = (P_1 - P_3)^2$, A is a normalization factor and b is an experimentally derived parameter. For very small values of t (θ), Coulomb scattering starts to dominate over hadronic scattering, in the region where the two are about equal the two amplitudes interfere with each other. Since the Coulomb amplitude is known from theory:

$$f_c(t) = 2\alpha\frac{G^2(t)}{t}e^{i\alpha\phi(t)} \tag{24}$$

where α and $G(t)$ come from experiment and $\phi(t)$ is derived from theory. The nuclear amplitude is written as

$$f_n(t) = \frac{\sigma_{\text{total}}}{4\pi}(\rho + i)e^{-bt/2} \tag{25}$$

where ρ is the ratio of real to imaginary parts of the scattering amplitude. By measuring differential cross section into the region where Coulomb scattering dominates and out to the region where nuclear scattering dominates, the value of ρ and σ_{total} can be fit, giving the total cross section and ratio of real to imaginary parts of the amplitude.