Physics 4213/5213
Lecture 7

1 Rotational Invariance and Angular Momentum

The homogeneity of space, has been shown to lead to the conservation of linear momentum. Here the isotropy (rotational invariance) of space will be shown to lead to the definition of an angular momentum operator and in the case where there are no external torques, it is conserved.

To derive the rotation operator and the form of the angular momentum operator, start with an arbitrary wave-function $\psi(\vec{x})$ and apply a rotation about a fixed axis (the standard axis to pick is the z-axis). The rotation is defined as:

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= y\cos\theta - x\sin\theta. \end{aligned} \tag{1}$$

As in the previous cases, the transformation is carried out in infinitesimal steps—this is doable since the transformation is continuous and in particular continuous from the identity transformation. The transformation then becomes:

$$\begin{aligned} x' &= x + y\delta\theta \\ y' &= y - x\delta\theta. \end{aligned} \tag{2}$$

The wave-function then becomes:

$$\psi(\vec{x}) \xrightarrow{R_z} \psi(x + y\delta\theta, y - x\delta\theta, z) \tag{3}$$

As in the case of translational invariance, the wave function is expanded about the point $\delta\theta = 0$. This gives, keeping only first order terms:

$$\psi(x + y\delta\theta, y - x\delta\theta, z) \approx \psi(\vec{x}) + \delta\theta\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\psi(\vec{x}), \tag{4}$$

where as in the previous lecture, the derivatives are related to the momentum operators through:

$$p_x = -i\frac{\partial}{\partial x} \qquad p_y = -i\frac{\partial}{\partial y}. \tag{5}$$

Substituting from equation 5 into equation 4, the rotation operator is seen to be

$$R = 1 + i\delta\theta L_z \qquad L_z = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) = (xp_y - yp_x) \tag{6}$$

where L_z is the z-component of the angular momentum. So finally, if the system is invariant to rotations $[R, H] = 0$ then the angular momentum is conserved $[L_z, H] = 0$.

The rotation operator above can be generalized to a rotation about any axis, which is then given by:

$$R = \left(1 + i\delta\vec{\theta}\cdot\vec{L}\right) \tag{7}$$

where $\vec{\theta}$ are the rotation angles about the three axis—these are typically taken as the Euler angles. To now go to finite rotations, an infinite number of infinitesimal rotations are applied:

$$\left.\begin{array}{r}\lim_{n\to\infty}(1-\delta\vec{\theta}\cdot\vec{L})^n=\psi(\vec{x}')\\ \lim_{n\to\infty}(1-\delta\vec{\theta}\cdot\vec{L})^n=e^{-i\vec{\theta}\cdot\vec{L}}\psi(\vec{x})\end{array}\right\}\Rightarrow U(\vec{\theta})\psi(\vec{x})=\psi(\vec{x}') \qquad U(\vec{\theta})=e^{-i\vec{\theta}\cdot\vec{L}}. \tag{8}$$

Notice that $U(\vec{\theta})$ is a unitary operator $U^{-1}(\vec{\theta})=U^{\dagger}(\vec{\theta})$ since $\vec{L}$ is Hermitian $\vec{L}=\vec{L}^{\dagger}$.

The rotation operators form a group in that they satisfy all the properties stated in the last lecture. The $\vec{L}$ are called the generators of the group; these are the quantities that generate the infinitesimal rotations. Their properties define all the properties of the group. In the case of the angular momentum operators, they are known to not commute with each other:

$$[L_x, L_y]=iL_z, \tag{9}$$

so that this is an example of an non-Abelian group; momentum is an Abelian group. The fact that the components do not commute, implies that only one of the components is measurable.

Next the total angular momentum is examined. This is given by $L^2=L_x^2+L_y^2+L_z^2$ and using the relations for the components of the angular momentum and the commutation relation in equation 9, the total angular momentum is shown to commute with all the components:

$$[L^2, L_i]=0. \tag{10}$$

Given the commutation relations for the angular momentum, the quantities used to label a state are L^2 and L_z.

The eigenvalues of L^2 are given by $L^2|l,m\rangle=l(l+1)|l,m\rangle$ while for L_z the eigenvalues are given by $L_z|l,m\rangle=m|l,m\rangle$. Also note that for a given value of angular momentum, there are $2l+1$ possible values of L_z; this is due to only one component being measurable, the quantizing of angular momentum and that there are both positive and negative values for L_z.

2 Addition of Angular Momentum

As in classical mechanics, if a system is composed of several particles of different angular momentum, the total angular momentum is given by the vector sum of all angular momenta. In quantum mechanics the sum is not as straight forward. This is due to the fact that the three components don't commute and therefore cannot be measured simultaneously. Given that only the total angular momentum and one of the components (the z component) can be measured simultaneously, the total angular momentum will have a range of values. The z component of the angular momentum is arrived at by simply adding the individual z components, while the total angular momentum has integer step values between l_1+l_2 and $|l_1-l_2|$ for a two particle system. For a multi-particle system, two particles are taken to start with, then a particle is added until all particles have been included.

In many cases the probability to be in a given angular momentum state is desired. This can be arrived at by the following, again two particles are used; the wave function is given by:

$$|l_1,m_1;l_2,m_2\rangle=\sum_{l=|l_1-l_2|}^{l_1+l_2}C_l|l,m\rangle, \qquad m=m_1+m_2. \tag{11}$$

Selecting out one of the states gives:

$$\langle l', m | l_1, m_1; l_2, m_2 \rangle = C_l \tag{12}$$

This gives the formal value of the coefficient C_l—these are referred to as Clebsch-Gordon coefficients. To get the numeric value requires a bit of work. This will be left to a course in quantum mechanics. The easiest way to arrive at the values is to look them up in a table. The tables are arranged by the value of the two angular momentum that are being added. The columns give the summed angular momentum and the z component of the summed angular momentum. The rows give the values of the z component of the angular momentum of each particle. The coefficients are given in the center of the table:

$$\begin{array}{ccccc} & & L & L & \cdots \\ l_1 & l_2 & M & M & \cdots \\ m_1 & m_2 & C_{LM}^{m_1 m_2} & C_{LM}^{m_1 m_2} & \cdots \\ m_1 & m_2 & C_{LM}^{m_1 m_2} & C_{LM}^{m_1 m_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \tag{13}$$

2.1 Spin and Total Angular Momentum

Particles can carry intrinsic angular momentum. This angular momentum, for point particles, is called spin (S). This acts just like the angular momentum described in the previous sections (orbital angular momentum) and can be added to it. The sum of spin and orbital angular momentum is denoted by J the total angular momentum. Note, all these quantities are the same, they are all angular momenta. They should all be treated the same and thought of in terms of quantum mechanics and not treated classically. Treating spin classically can lead to confusion.

3 Outline

1. Rotational Invariance
   (a) Show that it leads to angular momentum
   (b) Gives conservation of angular momentum
   (c) Unitary operator and generators of the group
2. Commutator
3. Spin, orbital and total angular momentum
4. Addition of angular momentum
5. Clebsch-Gordon coefficients