# Physics 4213/5213
# Lecture 6

## 1 Introduction–Symmetries in Physics

The use of symmetries in physics is useful in extracting information from experiment to build up theories. Symmetries are intimately connected to conservation laws through the use of Noether's theorem, which states that every symmetry of nature yields a conservation law. Conservation laws impose restrictions on the types of theories that can be built and also limit the quantities that have to be explored in experiment.

This and the next few lectures will discuss the use of symmetry principles in particle physics and how they can be used to determine whether a reaction will or will not occur and under what interaction the reaction is possible. These will add to the conservation laws that have already been discussed. This lecture starts by discussing what is meant by a symmetry principle, its connection to conservation laws and its connection to the mathematical theory of groups.

## 2 Symmetries and Group Theory

The first question that needs to be explored is, what is a symmetry. A symmetry specifies how physical quantities transform. For instance, the isotropy of space specifies that irrespective of what direction is observed, the laws of physics do not change. This leads to the conservation of angular momentum.

All symmetries are associated with specific transformations that leave the system invariant. Therefore a transformation $G$ carries the system into itself. Obviously, there must be a transformation $G'$ that takes the system to its original state and one that leaves the system unchanged $\mathbf{I}$. Finally, transformations can be combined that also lead to the system remaining invariant. All these describe the mathematical properties of groups:

1. *Closure.* If $G_1$ and $G_2$ are symmetry operations, then $G_3 = G_1 \cdot G_2$ is also a symmetry operation.

2. *Identity.* An identity element exists such that $G_1 \cdot \mathrm{I} = \mathrm{I} \cdot G_1 = G_1$.

3. *Inverse.* For every element $G_1$ there exists an inverse such that $G_1^{-1} \cdot G_1 = \mathrm{I}$.

4. *Associativity.* $G_1(G_2 \cdot G_3) = (G_1 \cdot G_2)G_3$.

In general the elements of a group do not commute. If they do commute, then the group is referred to as an *Abelian* group, if not, then it is a *non-Abelian* group.

Groups and therefore symmetries can be split into those that are continuous and those that are discreet. Examples are rotations for continuous and reflections for those that are discreet.

# 3 Invariance under Symmetry Operations

The invariance of a system under a symmetry operation implies that the system does not change. As an example, consider the invariance of the Lagrangian under translation—translations form a continuous group. For the Lagrangian to remain invariant under translations, it must satisfy the following relation:

$$L(x + \delta(x)) = L(x) + \delta L(x) = L(x) \Longrightarrow \delta L(x) = 0, \tag{1}$$

in one dimension and can trivially be expanded to three. The condition given in equation 1 above can be expressed in terms of derivatives of the Lagrangian as:

$$\delta L = \frac{\partial L}{\partial x}\delta x = 0 \Rightarrow \frac{\partial L}{\partial x} = 0. \tag{2}$$

Using the Euler-Lagrange equations leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{3}$$

where the last expression states that the canonical momentum is a constant. That is, that translations in space give momentum conservation. (*Note that this was done for a single particle in one dimension so the condition that there are no external forces is imposed. In general any number of particles can exist in any number of dimension, but again momentum is conserved only in those directions where there are no external forces.*)

## 3.1 Invariance of the Wave Function

Beyond classical systems, symmetry principles can also be used in quantum mechanics. Typically, the conservation laws that come from these symmetries are given in terms of commutation relations with the Hamiltonian. To see where this comes from, start with the Schrodinger equation:

$$i\frac{\partial}{\partial t}\psi_s(t) = H\psi_s(t), \tag{4}$$

where $\psi_s(t)$ denotes the Schrodinger representation—recall that in the Schrodinger representation, the time dependence is kept with the wave function while in the Heisenberg representation, the time dependence is kept with the operators. The wave function can be written as the product of the time translation operator and the wave function at a fixed time:

$$\psi_s(t) = e^{-i(t-t_0)H}\psi_H(t_0) = T(t, t_0)\psi_H(t_0). \tag{5}$$

The time dependence of any physically measurable quantity $Q$, is given by the expectation value of the operator associated with that quantity. The time dependence in this calculation can be kept either with the wave-function (Schrodinger representation) or with the operator (Heisenberg representation) with both methods giving the same result:

$$\langle q \rangle = \int \psi_s^*(t) Q_s \psi_s(t) dV = \int \psi_H^*(t_0) T^\dagger(t, t_0) Q_s T(t, t_0) \psi_H(t_0) dV. \tag{6}$$

In the second integral, the operators can be combined into a single time dependent operator $Q_H = T^{-1}(t,t_0)Q_sT(t,t_0)$, which is just the Heisenberg representation of the operator $Q$. The equation of motion of that operator can be derived by taking the total time derivative of $Q_H$:

$$i\frac{dQ_H}{dt} = i\left(\frac{dT^{-1}(t,t_0)}{dt}Q_sT(t,t_0) + T(t,t_0)Q_s\frac{dT^{\dagger}(t,t_0)}{dt}\right) = [Q_H, H], \tag{7}$$

where the last equality comes from taking the derivative of the time translation operator—note that the Hamiltonian is hermitian $H^{\dagger} = H$. Therefore, if the operator commutes with the Hamiltonian the operator is constant in time and therefore represents a conserved quantity; a constant of the motion.

Consider again spatial translations, except now this is done for the wave-function. As before start with infinitesimal translations:

$$\psi(x) \to \psi(x+\delta x) = \psi(x) + \delta x\frac{\partial\psi(x)}{\partial x} = \left(1 + \delta x\frac{\partial}{\partial x}\right)\psi(x) = D\psi(x), \tag{8}$$

where $D$ is the translation operator. Note that that the momentum operator is included in the displacement operator:

$$p_x = -i\frac{\partial}{\partial x} \Longrightarrow D = 1 + i\delta x p_x \tag{9}$$

To push this a little further and see how a finite translation can be arrived at, apply an infinite number of infinitesimal translations:

$$\left.\begin{array}{r}\lim_{n\to\infty} D^n\psi(x) = \lim_{n\to\infty}(1+ip\delta x)^n\psi(x) = \psi(x+\Delta x)\\ \lim_{n\to\infty} D^n\psi(x) = \lim_{n\to\infty}(1+ip\delta x)^n\psi(x) = e^{ip\Delta x}\psi(x)\end{array}\right\} \Longrightarrow e^{ip\Delta x}\psi(x) = \psi(x+\Delta x), \tag{10}$$

where $\Delta x = \lim_{n\to\infty} n\delta x$—*notice that $D$ is unitary; $DD^{\dagger} = \mathbf{1}$ and $p$ is referred to as the generator of the transformation (group).* Finally, if the Hamiltonian is independent of spatial translation, then

$$\left[e^{ip\Delta x}, H\right] = 0 \qquad \Rightarrow \qquad [p, H] = 0, \tag{11}$$

this follows by applying an infinite number of infinitesimal translations. Equation 11 implies that the momentum is a constant of the motion.