Physics 4213/5213
Lecture 2

1 Introduction

The first lecture covered, the particles that are believed to be elementary. That is those particles that are fundamental to our understanding of the field. These particles are classified into two groups, those that create matter and those that are the quanta of the field. The matter particles are all Fermions and are further broken up into quarks, which have strong interactions and leptons that do not, see table 1. The quarks are the constituents of hadrons, such as protons, neutrons and pions. Two types of hadrons exist, those that are Fermions are referred to as baryons and those that are bosons are referred to as mesons.

Fermion Families			Electric Charge
u	c	t	$2/3$
d	s	b	$-1/3$
e	μ	τ	-1
ν_e	ν_μ	ν_τ	0

Table 1: Table of the known elementary particles divided into families of fermions. The electric charge of each component member of the family is also given in terms of the proton charge.

In addition to those particles that make up matter, the particles that carry each of the forces were also given. These are required by both quantum mechanics and special relativity—energy is absorbed in discreet quanta and the speed at which a signal can propagate is limited by the speed of light. These particles are referred to as gauge particles and are given in table 2

Interaction	Quanta	Mass
Electromagnetic	γ	0
Weak	$W^{\pm}$, Z^0	82, 92 GeV
Strong	gluon (8)	0

Table 2: This list the interactions that are responsible for all particle interactions. Note that the gravitational interaction is ignored since it does not participate at any significant level in elementary particle interactions and it can not be quantized.

This lecture will look the charged leptons and give a qualitative introduction to electromagnetic interactions.

2 Properties of the Charged Leptons

The electron is simplest object to study, since it is the lightest charged particle and is believed to be a point; that is it has no size. Being the lightest charge particle, it has a mass of 0.511 MeV, and recalling that charge must be conserved, the electron cannot decay, it therefore has an infinite lifetime —recall that Maxwell's equations impose the condition that the net current through any closed surface must be equal to the change in charge within that volume:

$$\vec{\nabla} \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \partial^{\mu} J_{\mu} = 0 \tag{1}$$

this is the statement of charge conservation.

The size of the electron can be determined by scattering electrons off electrons and electrons off positrons. The experimental angular distribution (differential cross section) and the total collision probability (total cross section) are then compared to a theoretical calculation of point particles scattering off each other. From this method, the electron is found to have a radius of less than 1.2×10^{-17} cm. This should be compared to the size of the proton which is about 10^{-13} cm and the size of the hydrogen atom which is about 10^{-8} cm.

The electron is known to have a spin (intrinsic angular momentum) of $\frac{1}{2}\hbar$ making it a Fermion. Recall that the wave-function for two identical Fermion must be anti-symmetric under particle exchange ($\psi(1,2) = -\psi(2,1)$), this property will become important later on. From the angular momentum, the magnetic dipole moment of the electron can be determined. Recall from magnetostatics that the magnetic dipole moment of a loop of wire is given by $\mu = I \cdot A$, where I is the current and A is the area of the current loop. By taking the area to be an effective area, this formula can be extended to any shape. If the electron is assumed to form a current loop with an effective radius of r_{eff} the magnetic moment is given by (r_{eff} can be taken as zero in the end):

$$\mu = I \cdot A = \frac{ev}{2\pi r_{\text{eff}}} \pi r_{\text{eff}}^2. \tag{2}$$

The velocity can be converted to the angular momentum as $L = mvr_{\text{eff}}$. This leads to:

$$\mu = \frac{eL}{2\pi m_{\text{e}} r_{\text{eff}}^2} \pi r_{\text{eff}}^2 = \frac{e\hbar}{4m_{\text{e}}} \tag{3}$$

where the angular momentum of the electron $L = \hbar/2$ is substituted in the last step. This value turns out to be a factor of two too small. The difference is given by the Lande g-factor which is given by $g = 2$, therefore the electron magnetic moment is

$$\mu = g\frac{e\hbar}{4m_{\text{e}}} = \frac{e\hbar}{2m_{\text{e}}}. \tag{4}$$

As a final comment, the value of g in quantum mechanics comes from atom experiments where the atom is placed in a magnetic field and electron spin flip transitions are observed. In relativistic quantum mechanics, the value of g is predicted from the theory.

At this point just about everything that is known about the electron has been given. In fact, everything that was just stated about the electron is true about the muon and the tau lepton except that the masses are larger. Table 3 gives the properties of the three charge leptons. One difference between the electron and the other two leptons is that they can decay to lighter particles and still conserve charge.

Yet, these leptons can not decay into any arbitrary particles. For instance the following decays should occur

$$\mu \to e + \gamma$$
$$\tau \to \mu + \gamma,$$

Lepton	Mass	Magnetic Moment
e^-	0.511 MeV	5.8×10^{-11} MeV/T
μ^-	105.66 MeV	2.8×10^{-13} MeV/T
τ^-	1.777 GeV	1.7×10^{-14} MeV/T

Table 3: The properties of the charged leptons.

since they conserve charge, energy and angular momentum. Yet these reactions have never been observed. This leads to the postulation of a new conserved quantity called lepton number—when weak interactions are discussed, it will be shown that there are in fact three different lepton numbers one associated with each of the three lepton families.

3 Electrodynamics of Charged Leptons

The next question that needs to be answered, is how do charged point particles (leptons) interact—only electromagnetic interactions will be considered for now, but the same arguments will be used for other interactions later on. In classical electrodynamics the interaction of two particles can be treated as two currents interacting through their electric and magnetic fields or through the scalar and vector potentials they generate:

$$\nabla^2 \vec{\mathbf{A}} - \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = \vec{\mathbf{J}} \qquad \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = \rho. \tag{5}$$

For simplicity, the relativistic four-vector notation will be introduced. This notation is centered around the scalar product of two vectors:

$$x^\mu x_\mu = t^2 - x^2 - y^2 - z^2, \tag{6}$$

which implies that a sign difference exists between the up index and the down index in the space part of the vector. The standard definition is

$$x^\mu = (t, x, y, z) \qquad x_\mu = (t, -x, -y, -z) \tag{7}$$

In the case of the derivatives, the signs are reversed

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = (\partial_t, \partial_x, \partial_y, \partial_z) \qquad \frac{\partial}{\partial x_\mu} = \partial^\mu = (\partial_t, -\partial_x, -\partial_y, -\partial_z). \tag{8}$$

Using this notation, the equation for the potentials is given by:

$$\partial^\mu \partial_\mu A^\nu = J^\nu \tag{9}$$

where a sum is implied over up-down like indices.

The advantage of using this notation, is to recall that in quantum mechanics space derivatives correspond to momenta, while time derivatives correspond to energies—this allows the use of the energy-momentum 4-vector. Since the derivatives correspond to how the fields change from position to position, the derivatives can be thought of as 4-momentum differences:

$$\partial^\mu \Longrightarrow -i(p_{f\mu} - p_{i\mu}), \tag{10}$$

where the 4-momentum differences corresponds to the current since it generates the field or it can be defined as the 4-momentum of the field. But in quantum mechanics the field is to be thought of as a collection of photons, the 4-momentum is that of the photons:

$$q^\mu = (p_f^\mu - p_i^\mu). \tag{11}$$

Equation 9 can then be written in terms of a current and a photon:

$$(p_f^\mu - p_i^\mu)(p_{f\mu} - p_{i\mu})A^\nu = J^\nu \Longrightarrow A^\nu = \frac{J^\nu}{q^2}, \tag{12}$$

where the vector potential now describes the scattering amplitude—notice that the amplitude decreases with increasing momentum transfer.

The current itself can be thought of as a plane wave multiplied by the charge of the particle generating the field A^μ:

$$J = q_{\mathrm{e}} N_1 e^{iq^\mu x_{1\mu}}; \tag{13}$$

N_1 is a normalization constant, that for now can be taken as giving the correct units to the current. Next recall that the interaction of the field (particle that generates the field) with a second particle needs to be described by either a Hamiltonian or Lagrangian, that means that the interaction energy needs to be calculated. The interaction energy is given by $J_1^\mu A_\mu = J_1^\mu J_\mu^2/q^2 \propto q_{e_1} q_{e_2}/q^2$ Therefore, the amplitude for two charged particles scattering off each other contains three components:

1. The current for particle 1;
2. The current for particle 2;
3. A term that corresponds to a photon.

Since this corresponds to the scattering amplitude, the probability (differential cross section) is the magnitude squared and therefore it will be proportional to

$$\frac{d\sigma}{d\Omega} \propto \frac{q_e^4}{q^4} \propto \frac{\alpha_{\mathrm{em}}^2}{q^4}. \tag{14}$$

This expression can be rewritten in terms of angular quantities as follows;

$$q^2 = (p_f - p_i)^\mu (p_f - p_i)_\mu = 2m_e^2 - 2(E_f E_i - |p_f||p_i|\cos\theta) \tag{15}$$

In the limit of very high energy scattering $E >> m$, this expression can be written as;

$$q^2 \approx 4|p_f||p_i|\sin^2\left(\frac{\theta}{2}\right) \tag{16}$$

so that finally the differential cross section is given by:

$$\frac{d\sigma}{d\Omega} \propto \frac{\alpha_{\mathrm{em}}^2}{\sin^4(\theta/2)} \tag{17}$$

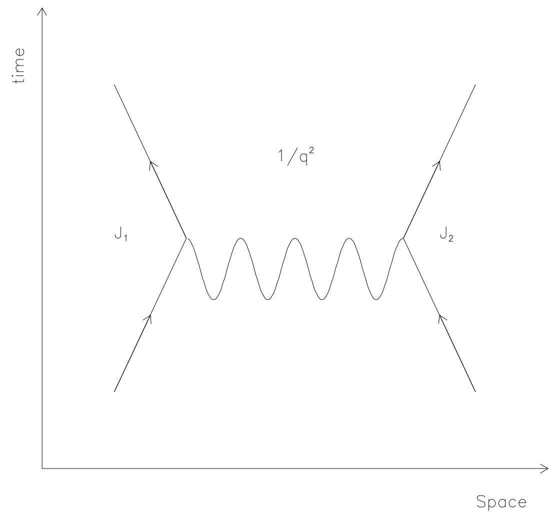

Figure 1: Feynman diagram for $e^- + e^- \to e^- + e^-$.

3.1 The Amplitude and the Coulomb Potential

From the information given above, the amplitude can be shown to be equal to the Coulomb potential. The attraction between two charge objects can be thought of as the exchange of an infinite number of photons with an infinite range of momenta. Therefore the amplitude can be integrate over all possible photon momenta (for simplicity do in one dimension):

$$\int \frac{q_e q_e}{q^2} e^{iq(x_2 - x_1)} dq = \frac{q_e q_e}{r}. \tag{18}$$

3.2 Feynman Diagram

Everything that was just stated can be put into a simple pictorial form called a Feynman diagram (these were invented independently by Stuckelberg during World War II). The Feynman is a space-time picture that describes a scattering process. In fact, these diagrams are intimately connected to a perturbation series expansion in quantum field theory and therefore correspond to a given scattering amplitude. From field theory rules can be developed to calculate these diagrams and for the most part for electrodynamics the rules are as given in section 3. Figure 1 show the Feynman diagram for two electrons scattering.

4 Outline

1. Properties of the electron
   (a) mass
   (b) charge
   (c) spin, magnetic moment
   (d) size
2. Electromagnetic interaction
   (a) EM current and the propagator.
   (b) Vertex factor
3. Feynman diagrams
   (a) Particles and anti-particles
   (b) Attraction and repulsion
   (c) Estimate ratios