7.4 The Photon

Ailin Deng

12/3/2012

Maxwell's Equations (Classical)

$$\vec{\nabla}\bullet\vec{E} = 4\pi\rho \qquad (1)$$

$$\vec{\nabla}\bullet\vec{B} = 0 \qquad (2)$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad (3)$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \qquad (4)$$

Introducing:

Field Strength Tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Four-vector

$$J^{\mu} = \left(c\rho, \vec{J}\right)$$

$$A^{\mu} = \left(V, \vec{A}\right)$$

Inhomogeneous:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

⇓

$$\vec{\nabla} \bullet \vec{E} = 4\pi\rho \qquad (1)$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \qquad (4)$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$J^\mu = \left(c\rho, \vec{J}\right)$$

---

covariant $$\partial_\mu [\ ] = \frac{1}{c}\frac{\partial}{\partial t}[\ ]^0 + \frac{\partial}{\partial x}[\ ]^1 + \frac{\partial}{\partial y}[\ ]^2 + \frac{\partial}{\partial z}[\ ]^3$$

contravariant $$\partial^\mu [\ ] = \frac{1}{c}\frac{\partial}{\partial t}[\ ]^0 - \frac{\partial}{\partial x}[\ ]^1 - \frac{\partial}{\partial y}[\ ]^2 - \frac{\partial}{\partial z}[\ ]^3$$

Exploiting the antisymmetry of $F^{\mu\nu}$, we can derive the

$$\text{Continuity Equation:} \quad \vec{\nabla}\bullet\vec{J} = -\frac{\partial\rho}{\partial t}$$

using $$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

$$0 \stackrel{?}{=} \partial_\nu(\partial_\mu F^{\mu\nu}) = \frac{4\pi}{c}\partial_\nu J^\nu$$

⇓

$$\partial_\nu J^\nu = 0$$

⇓

$$\frac{1}{\not{c}}\frac{\partial \not{c}\rho}{\partial t} + \vec{\nabla}\bullet\vec{J} = 0$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$J^\mu = \left(c\rho, \vec{J}\right)$$

Homogeneous:

$$\vec{\nabla}\bullet\vec{B}=0 \qquad (2) \quad \Rightarrow \quad \vec{B}=\vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0 \qquad (3) \quad \Rightarrow \quad \vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}=-\nabla V$$

$$F^{\mu\nu}=\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$$

$$A^\mu=\left(V,\vec{A}\right)$$

Review:

Inhomogeneous:
$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

Homogeneous:
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

We get,
$$\partial_\mu\left(\partial^\mu A^\nu\right) - \partial_\mu\left(\partial^\nu A^\mu\right) = \frac{4\pi}{c} J^\nu$$

Inhomogeneous:
$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu$$

Homogeneous: $$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Nice?

But actually there's a **Defect!**

V and $\vec{A}$ is not uniquely determined!

With a new potential $A_\mu{}' = A_\mu + \partial_\mu \lambda$ (Gauge Transformation)

$\partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime} = F^{\mu\nu}$ will do just as well!

Since $\partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Can't determine the vector $A^{\mu} = \left(V, \vec{A}\right)$?

Extra Constraint!

$\partial_{\mu} A^{\mu} = 0$ (Lorentz Condition)

Inhomogeneous: $\partial_{\mu}\partial^{\mu} A^{\nu} - \partial^{\nu}\partial_{\mu} A^{\mu} = \frac{4\pi}{c} J^{\nu}$

$$\Box A^{\nu} = \frac{4\pi}{c} J^{\nu}$$

with $\Box = \partial_{\mu}\partial^{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

But there still exists the gauge $\Box \lambda = 0$

$A^{\mu} = \left(V, \vec{A}\right)$ Still Not Determined!

One constraint is not enough? Add another one!

In empty space, $J^{\mu} = 0$, we pick

$A^{0} = 0$ (extra constraint)

$\partial_{\mu} A^{\mu} = 0$ (Lorentz Condition)

$\vec{\nabla} \bullet \vec{A} = 0$ (Coulomb gauge)

$J^{\mu} = \left(c\rho, \vec{J}\right)$ $A^{\mu} = \left(V, \vec{A}\right)$

Inhomogeneous: $\boxed{\Box A^{\nu} = \frac{4\pi}{c} J^{\nu}}$ with $\Box = \partial_{\mu}\partial^{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

Now we're prepared to get to the wave function of

Free Photon!

---

free photon: $J^{\mu} = 0$

$\boxed{\Box A^{\nu} = \frac{4\pi}{c} J^{\nu}}$ becomes $\boxed{\partial_{\mu}\partial^{\mu} A^{\nu} = 0}$

we have plane-wave solution

$$A^{\mu}(x) = a\,\mathbf{exp}\left(-\frac{i}{\hbar} p \bullet x\right) \boxed{\varepsilon^{\mu}(p)}$$

← **Polarization vector**

plane-wave solution

(Inhomogeneous Maxwell's Eq)

$$A^{\mu}(x) = a\,\mathbf{exp}\left(-\frac{i}{\hbar}\,p \bullet x\right)\quad \varepsilon^{\mu}(p) \longrightarrow \boxed{\partial_{\mu}\partial^{\mu}A^{\nu} = 0}$$

we get $\boxed{p^{\mu}p_{\mu} = 0}$ or $E = |\vec{p}|c$

---

plane-wave solution

(Lorentz Condition)

$$A^{\mu}(x) = a\,\mathbf{exp}\left(-\frac{i}{\hbar}\,p \bullet x\right)\quad \varepsilon^{\mu}(p) \longrightarrow \boxed{\partial_{\mu}A^{\mu} = 0}$$

we get $\boxed{\varepsilon^{\mu}(p)\;p_{\mu} = 0}$

plane-wave solution

$$A^{\mu}(x) = a\,\mathbf{exp}\left(-\frac{i}{\hbar}\,p\bullet x\right)\quad \varepsilon^{\mu}(p)$$

Lorentz Condition $\boxed{\varepsilon^{\mu}(p)\;p_{\mu} = 0}$

Using Coulomb gauge, $A^{0} = 0$

$$A^{0}(x) = a\,\mathbf{exp}\left(-\frac{i}{\hbar}\,p\bullet x\right)\quad \varepsilon^{0}(p) \Rightarrow \boxed{\varepsilon^{0}(p) = 0}$$

with $\varepsilon^{\mu}(p)\ p_{\mu} = 0$ and $\varepsilon^{0}(p) = 0$ ⇒ $\vec{\varepsilon} \bullet \vec{p} = 0$

Meaning the polarization three-vector $\vec{\varepsilon}$ is a perpendicular to the direction of propagation.

$\therefore$ free photon is transversely polorized.

If $\vec{p}$ is in z-direction, we can choose

$\overrightarrow{\varepsilon^{(1)}} = (1,0,0)$

$\overrightarrow{\varepsilon^{(2)}} = (0,1,0)$

Thank you