①

covariant $\partial_\mu[\,] = \frac{\partial}{c\partial t}[\,]^0 + \frac{\partial}{\partial x}[\,]^1 + \cdots$ Heads up

contravariant $\partial^\mu[\,] = \frac{\partial}{c\partial t}[\,]^0 - \frac{\partial}{\partial x}[\,]^1 - \cdots$

Maxwell Equation :

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ ① $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$ ②

$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$ ③ $\qquad$ $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$ ④

We introduce the 'field strength tensor' $F^{\mu\nu}$

Field strength tensor $\quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

and the

four-vector : $J^\mu = (c\rho, \vec{J})$

INHOMO

so that two inhomogeneous Maxwell equations can be written into one neat tensor notation

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu \quad \Rightarrow \quad \begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \end{cases}$$

zero here indicates the first column of $F^{\mu\nu}$

Eg.① $\partial_\mu F^{\mu 0} = \frac{4\pi}{c}J^0$ with $\partial_\mu = \frac{1}{c\partial t}[\,]^0 + \frac{1}{\partial x}[\,]^1 + \frac{1}{\partial y}[\,]^2 + \frac{1}{\partial z}[\,]^3$

explicitly,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{4\pi}{\cancel{c}}\cdot\cancel{c}\rho$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

1 here indicates the 2^{nd} column of $F^{\mu\nu}$

Eq. ④ $\partial_\mu F^{\mu 1} = \frac{4\pi}{c} J^1$

explicitly

$$\frac{1}{c}\frac{\partial}{\partial t}(-E_x) + \frac{\partial}{\partial x}\cdot 0 + \frac{\partial}{\partial y} B_z + \frac{\partial}{\partial z}(-B_y) = \frac{4\pi}{c} J_x$$

$$\Rightarrow \left[-\frac{1}{c}\frac{\partial E}{\partial t} + \vec{\nabla}\times\vec{B}\right]_x = \frac{4\pi}{c} J_x$$ which is the x component of ④

We can work out the y, z component in the same way,

write on board $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$

Continuity Eq. $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$

The 'continuity equation' expressing local conservation of charge is also included in this '$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$' notation, exploiting the antisymmetry of $F^{\mu\nu}$

$$\frac{4\pi}{c} J^\nu = \partial_\mu F^{\mu\nu}$$

$$\frac{4\pi}{c}\cdot \partial_\nu J^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0$$

?

$$\sum_{\nu=0}^{3}\sum_{\mu=0}^{3} \partial_\nu \partial_\mu F^{\mu\nu} = \partial_0\partial_0 F^{00} + \partial_1\partial_1 F^{11} + \cdots \underbrace{\partial_1\partial_2 F^{21} + \partial_2\partial_1 F^{12}}_{0 \text{ for } F^{21} = -F^{12}} + \cdots$$

$F^{00} \to 0$, $F^{11} \to 0$: these are zeros

and we can always pair them up and get 0

$$\partial_\nu J^\nu = 0 \Rightarrow \frac{1}{c}\frac{\partial \rho c}{\partial t} + \frac{\partial}{\partial x} J_x + \frac{\partial}{\partial y} J_y + \frac{\partial}{\partial z} J_z = 0.$$

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \quad !!$$

Homo

We've dealt with the inhomogeneous equations.
Now we start working on the homogeneous ones
Let's make some observation before we actully work on the tensor and the vector

② $\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A}$ ②*

curl of a vector potential

Plug ②* into ③ we have

OBSERVE: ③ $\vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0 \quad \Rightarrow \quad \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\nabla V$ ③*

which means $\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ can be written as the gradient of a scalar potential V

Now that we get these results, we'll continue our work on the tensor and vector.

Introducing a new four-vector $A^\mu = (V, \vec{A})$

②* ③* become $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Eg. ②* $\quad F^{12} = (\partial^1 A^2) - (\partial^2 A^1)$

$$\Rightarrow \quad -B_z = \left(-\frac{\partial}{\partial x}A_y\right) - \left(-\frac{\partial}{\partial y}A_x\right)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{\partial}{\partial x}A_y - \frac{\partial A_x}{\partial y}$$

which is $(\vec{B})_z = (\vec{\nabla}\times\vec{A})_z$ (z component of)

Eg ③*
$$F^{01} = (\partial^0 A^1) - (\partial^1 A^0)$$
$$\Rightarrow \quad -E_x = \frac{1}{c}\frac{\partial}{\partial t} A_x - \left(-\frac{\partial}{\partial x} V\right)$$

which is the x component of ③*
$$(\vec{E})_x = \left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \nabla V\right)_x$$

write on board $\quad \cancel{\partial_\mu F^{\mu\nu}} = F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Take a Review

Inhomo : $\quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \quad$ ⑤

Homo : $\quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad$ ⑥.

Plug ⑥ into ⑤, we have

interchange the order

Inhomo $\quad \partial_\mu(\partial^\mu A^\nu) - \partial_\mu(\partial^\nu A^\mu) = \frac{4\pi}{c} J^\nu$

or

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = \frac{4\pi}{c} J^\nu \quad ⑤^*$$

write on board

Every thing seems to be nice, but there's actually a Defections ! in the (5)* notation ! ...

Defection : V and $\vec{A}$ is not uniquely determined !

When we have (6) $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

with a New Potential $A_\mu' = A_\mu + \partial_\mu \lambda$ with λ any function of position time (with $\lambda(\vec{x}, t)$)

$\partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime} = F^{\mu\nu}$ will do just as well (!

Since $\partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

In the interest of time, we're not going to prove it here, but it's in the note.

Proof:

$$\begin{aligned}
&\partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime} \\
&= \partial^\mu (g^{\nu\rho} A_\rho') - \partial^\nu (g^{\mu\sigma} A_\sigma') \\
&= \partial^\mu g^{\nu\rho} (A_\rho + \partial_\rho \lambda) - \partial^\nu g^{\mu\sigma} (A_\sigma + \partial_\sigma \lambda) \\
&= \partial^\mu A^\nu + \cancel{\partial^\mu \partial^\nu \lambda} - \partial^\nu A^\mu - \cancel{\partial^\nu \partial^\mu} \lambda \\
&= \partial^\mu A^\nu - \partial^\nu A^\mu
\end{aligned}$$

Such A change of potential is called $\Rightarrow$ a gauge Transformation

We exploit this gauge freedom to impose an Extra Constraint on A^μ :

$$\partial_\mu A^\mu = 0 \qquad \text{(Lorentz Condition)}$$

write one board !

can be simplied to be ↙ write on board.

Then ⑤* $\Rightarrow$ $\Box A^\nu = \frac{4\pi}{c} J^\nu$ ⑤**

where $\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ (d'Alembertian).

(which is the relativistic extension of ∇^2, it's called

With the Lorentz Con., further gauge is still possible,

if $\Box \lambda = 0$, $\partial_\mu A^{\mu'} = 0$

Again, you can find the proof in the note.

Proof: $A_\mu' = A_\mu + \partial_\mu \lambda$

$$\partial_\mu A^{\mu'} = \partial_\mu g^{\mu\nu} A_\nu' = \partial_\mu g^{\mu\nu}(A_\nu + \partial_\nu \lambda) = \partial_\mu A^\mu + \underbrace{\partial_\mu \partial^\mu \lambda}_{= 0 \text{ if } \Box\lambda = 0}$$

$\therefore \partial_\mu A^{\mu'} = \partial_\mu A^\mu = 0$ (with $\Box \lambda = 0$)

A^μ (not determined) < ~~live with that~~ / $\boxed{\text{extra constrains}}$ (Both are used)

In empty space $J^\mu = 0$, we pick $A^0 = 0$ ($V = 0$)

The Lorentz Constraint $\partial_\mu A^\mu = 0$ $\Rightarrow$ $\vec{\nabla} \cdot \vec{A} = 0$ (Coulomb gauge)

Finally, we can get to the wave function of free photons.

free photon : $J^\mu = 0$

(5)*# $\Box A^\mu = \frac{4\pi}{c} J^\mu \Rightarrow \Box A^\mu = 0$ (7)

we look for a plane wave solution

plane-wave solutions

$$A^\mu(x) = a\, e^{-\frac{i}{\hbar} p\cdot x} \underbrace{\epsilon^\mu(p)}_{\text{polarization vector (spin)}} \quad (8)$$

related to

Substituting, (8) into (7), we have

$$\partial^\nu \partial_\nu A^\mu = \partial^\nu \left(\partial_\nu a e^{-\frac{i}{\hbar} p_\mu x^\mu} \epsilon^\mu(p)\right)$$

$$= a\epsilon^\mu(p)\, \partial^\nu \left\{ \left(e^{-\frac{i}{\hbar} p_\mu x^\mu}\right) \cdot \left(-\frac{i}{\hbar} p_\mu\right) \cdot \boxed{\partial_\nu x^\mu} \right\}$$

$$= a\epsilon^\mu(p) \left(-\frac{i}{\hbar} p_\mu\right) \partial^\nu \left(e^{-\frac{i}{\hbar} p^\mu x_\mu}\right)$$

$$= a\epsilon^\mu(p) \left(-\frac{i}{\hbar} p_\mu\right)\left(-\frac{i}{\hbar} p^\mu\right) \left(e^{-\frac{i}{\hbar} p^\mu x_\mu}\right) \boxed{\left(\partial^\nu x_\mu\right)}$$

$$= a\epsilon^\mu(p) \left(-\frac{1}{\hbar^2}\right) \left(p_\mu p^\mu\right) e^{-\frac{i}{\hbar} p\cdot x}$$

So we have constraint on p_μ $\boxed{p_\mu p^\mu = 0}$ or $E = |\vec{p}|c$ $\left(m^2c^2 = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2 = 0\right)$ (9)

At the same time,

Substituting (8) into Lorentz Conditions $\partial_\mu A^\mu = 0$ and we have

$$\epsilon^\mu(p)\, p_\mu = 0 \quad \text{(Lorentz Condition)} \quad (10)$$

In Coulomb gauge, $A^0 = 0$

$A^0(x) = a e^{-\frac{i}{\hbar} p \cdot x} \epsilon^0(p) = 0 \quad \Rightarrow \epsilon^0(p) = 0.$ (Coulomb gauge).

Thus

$\vec{\epsilon} \cdot \vec{p} = 0$ meaning the polarization three-vector $\vec{\epsilon}$ is perpendicular to the direction of propagation.

$\therefore$ free photon is transversely polarized.

If $\vec{p}$ is in z-direction, we can choose

$\vec{\epsilon}^{(1)} = (1, 0, 0) \quad \vec{\epsilon}^{(2)} = (0, 1, 0)$

$\therefore$ free photon has only two spin orientations