# 7.1 The Dirac Equation

PHYS 5213
Nuclear and Particle Physics

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Schrödinger Equation (nonrelativistic):

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

this comes from the classical energy-momentum relation:

$$\frac{\vec{p}^{\,2}}{2m} + V = E$$

$$\vec{p} \rightarrow -i\hbar\nabla \qquad E \rightarrow i\hbar\frac{\partial}{\partial t}$$

now, in relativistic case:

$$E^2 = |\vec{p}|^2c^2 + m^2c^4$$

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or $$p^\mu p_\mu - m^2 c^2 = 0$$

where
$$p^\mu = \left( E/c \, , \vec{p} \right)$$

$$p_\mu = i\hbar \partial_\mu = i\hbar \frac{\partial}{\partial x^\mu}$$

$$\Downarrow$$

$$-\hbar^2 \partial^\mu \partial_\mu \psi - m^2 c^2 \psi = 0 \qquad \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

or $$\boxed{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi + \nabla^2 \psi = \left(\frac{mc}{\hbar}\right)^2} \rightarrow \text{Klein-Gordon Eq.}$$

$$(p^\mu p_\mu - m^2c^2) = (\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc)$$

or

$$(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 - m^2c^2 =$$

$$(\beta^0 p^0 - \beta^1 p^1 - \beta^2 p^2 - \beta^3 p^3 + mc)(\gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3 - mc)$$

$$(\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = \beta^\kappa \gamma^\lambda p_\kappa p_\lambda - mc(\beta^\kappa p_\kappa - \gamma^\lambda p_\lambda) - m^2c^2$$

$$= \beta^\kappa \gamma^\lambda p_\kappa p_\lambda - mc(\beta^\kappa - \gamma^\kappa) p_\kappa - m^2c^2$$

$$\beta^\kappa - \gamma^\kappa = 0$$

$$\beta^{\kappa} - \gamma^{\kappa} = 0$$

and $\quad p^{\mu} p_{\mu} = \beta^{\kappa} \gamma^{\lambda} p_{\kappa} p_{\lambda} = \gamma^{\kappa} \gamma^{\lambda} p_{\kappa} p_{\lambda}$

$$p^{\mu} p_{\mu} = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$$

$$\begin{aligned}
\gamma^{\kappa} \gamma^{\lambda} p_{\kappa} p_{\lambda} = {} & (\gamma^0)^2 (p_0)^2 + (\gamma^1)^2 (p_1)^2 + (\gamma^2)^2 (p_2)^2 + (\gamma^3)^2 (p_3)^2 \\
& + (\gamma^0 \gamma^1 + \gamma^1 \gamma^0) p_0 p_1 + (\gamma^0 \gamma^2 + \gamma^2 \gamma^0) p_0 p_2 \\
& + (\gamma^0 \gamma^3 + \gamma^3 \gamma^0) p_0 p_3 + (\gamma^1 \gamma^2 + \gamma^2 \gamma^1) p_1 p_2 \\
& + (\gamma^1 \gamma^3 + \gamma^3 \gamma^1) p_1 p_3 + (\gamma^2 \gamma^3 + \gamma^3 \gamma^2) p_2 p_3
\end{aligned}$$

$$(\gamma^0)^2 = 1, \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$$

$$(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = 0 \quad \text{for } \mu \neq \nu$$

$$\Rightarrow \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\{A, B\} = AB + BA \quad \leftarrow \text{anticommutator}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

or $$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$p^\mu p_\mu - m^2c^2 = (\gamma^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = 0$$

$\gamma^\lambda p_\lambda - mc = 0$ by convention.

$$\Rightarrow \boxed{i\hbar\gamma^\mu \partial_\mu \psi - mc\psi = 0} \leftarrow \text{Dirac Eq.}$$