presentation

Scattering process

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$1+2 \longrightarrow 3+4+\cdots+n$

initial state: $|\phi_1 \phi_2\rangle$: We prepare

final state : $\langle \phi_3 \phi_4 \cdots \phi_n|$: We obtain

ideal $|\psi\rangle_{real} \xrightarrow[T\to\infty]{} |\psi\rangle_{ideal}$

real experiment

the probability to obtain the final state

$$P = |\underbrace{\langle \phi_3 \phi_4 \cdots \phi_n}_{\text{future}} | \underbrace{\phi_1 \phi_2 \rangle}_{\text{past}}|^2$$

$$_{out}\langle p_3 p_4 \cdots p_n | k_1 k_2\rangle_{in} = \lim_{T\to\infty} \langle p_3 p_4 \cdots p_n | e^{-iH(2T)} | k_1 k_2\rangle$$

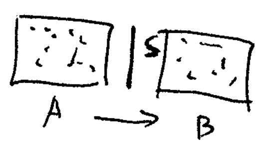
this is the definition of S-matrix : $_{out}\langle p_3 p_4 \cdots p_n | k_1 k_2\rangle_{in} \equiv \langle p_3 p_4 \cdots p_n | S | k_1 k_2\rangle$

Sometimes, to isolate the interaction parts
we define T-matrix $\quad S = \mathbb{I} + iT$

then the invariant matrix element can be defined as

$$\langle p_3 p_4 \cdots p_n | iT | k_1 k_2\rangle = (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \Sigma p_f)\, i\mathcal{M}(k_1, k_2 \to p_f)$$

$$\sigma = \frac{\text{Number of events}}{\rho_A l_A \rho_B l_B \, S}$$



For a single target (A) particle and many incident particles (B) with different impact parameter b

$$N = \int d^2b\, n_B P(b)$$

$$\sigma = \frac{N}{n_B N_A} = \frac{N}{n_B \cdot 1} = \int d^2 b\, P(b)$$

Straightforward from $P(b) \longrightarrow \sigma$

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(k) |k\rangle$$

and $|k\rangle = \sqrt{2E_k}\, a^\dagger_k |0\rangle$

Scattering cross section

$$\sigma = \frac{S\hbar^2}{4\sqrt{(P_1\cdot P_2)^2-(m_1m_2c^2)^2}}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(P_1+P_2-P_3\cdots-P_n)\prod_{j=3}^{n} 2\pi\delta(P_j^2-m_j^2c^2)\theta(P_j^0)$$

$$\frac{d^4P_j}{(2\pi)^4}$$

three kinematical constraints :

① mass shell

② outgoing energy is positive

③ energy. momentum conserved

$$\int d\Pi = \prod_f \left(\frac{d^3P_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}(P-\sum P_f)$$

relativistically invariant

n-body phase space

Performing the P_j^0 integral

$$\delta((P_j^0)^2-\vec{P}_j^{\,2}-m_j^2c^2)$$

then
$$\sigma = \frac{S\hbar^2}{4\sqrt{(P_1\cdot P_2)^2-(m_1m_2c^2)^2}}\int |\mathcal{M}|^2(2\pi)^4\delta^4(P_1+P_2-P_3\cdots-P_n)$$
$$\times\prod_{j=3}^{n}\frac{1}{2\sqrt{\vec{P}_j^{\,2}+m_j^2c^2}}\frac{d^3\vec{P}_j}{(2\pi)^3}$$

example : two-body Scattering in the CM frame

$$1+2\rightarrow 3+4$$

~~Simply~~ Simplify the denominator

where $\sqrt{(P_1\cdot P_2)^2-(m_1m_2c^2)^2} = \sqrt{(P_1^0P_2^0+\vec{P}_1\cdot\vec{P}_2)^2-(m_1m_2c^2)^2}$

by $P_j^0=\sqrt{|\vec{P}_j|^2+m_j^2c^2}$ $\Rightarrow$ $= \sqrt{(P_1^0P_2^0+|\vec{P}_1|^2)^2-m_1^2m_2^2c^4}$

$\cdots$

$$=(E_1+E_2)|\vec{P}_1|/c$$

the scattering cross section then reads:

$$\sigma = \frac{S\hbar^2 c}{64\pi^2(E_1+E_2)|\vec{P}_1|}\int |\mathcal{M}|^2 \frac{\delta^4(P_1+P_2-P_3-P_4)}{\sqrt{\vec{P}_3^{\,2}+m_3^2c^2}\sqrt{\vec{P}_4^{\,2}+m_4^2c^2}}\, d^3\vec{P}_3\, d^3\vec{P}_4$$

using $\delta^4(P_1+P_2-P_3-P_4) = \delta\left(\frac{E_1+E_2}{c} - P_3^0 - P_4^0\right)\delta(\vec{P}_3+\vec{P}_4)$

$$= \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{P}_1|}\int |\mathcal{M}|^2 \frac{\delta\left[(E_1+E_2)/c - \sqrt{|\vec{P}_3|^2+m_3^2c^2}\sqrt{\vec{P}_3^{\,2}+m_4^2c^2}\right]}{\sqrt{\vec{P}_3^{\,2}+m_3^2c^2}\sqrt{\vec{P}_3^{\,2}+m_4^2c^2}}\, d^3\vec{P}_3$$

In spherical coordinate

$$d^3\vec{P}_3 = p^2\,dp\,d\Omega \qquad \text{where } p = |\vec{P}_3|$$

then
$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{P}_1|}\int_0^\infty |\mathcal{M}|^2 \frac{\delta\left[(E_1+E_2)/c - \sqrt{p^2+m_3^2c^2} - \sqrt{p^2+m_4^2c^2}\right]}{\sqrt{p^2+m_3^2c^2}\sqrt{p^2+m_4^2c^2}}\, p^2\,dp$$

Using the same method while discussing decay formula

let $u = \sqrt{p^2+m_3^2c^2} + \sqrt{p^2+m_4^2c^2}$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1+E_2)^2}\frac{|\vec{P}_f|}{|\vec{P}_i|}$$ Differential cross section

small remark: $$|k_A k_B\rangle \propto \lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|k_A k_B\rangle_0$$

$$\lim_{T\to\infty(1-i\epsilon)} {}_0\langle \vec{P}_3\vec{P}_4\cdots\vec{P}_n| e^{-iH(2T)} |\vec{P}_1\vec{P}_2\rangle_0$$

$$\propto \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \vec{P}_3\vec{P}_4\cdots\vec{P}_n| T\left(\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right)|\vec{P}_1\vec{P}_2\rangle_0$$