Decays

Particle Physics Presentation

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Lifetime

In the case of decays, the item of greatest interest is the *lifetime* of the particle.

- We cannot to calculate the lifetime of any *particular* particle.
- The *average* lifetime (mean lifetime) τ.
- $\tau = \frac{1}{\Gamma_{tot}}$, where Γ_{tot} is the *total* decay rate.

Conclusion

In the case of decays, the item of greatest interest is the ~~lifetime~~ decay rate, Γ_{tot}, of the particle.

Decay rate

Definition

Decay rate Γ = The probability per unit time that a given particle will decay.

$N(t)$: The number of particles at time t.

$$
\begin{aligned}
dN &= N(t) - N(t=0) < 0 \\
dN &= -\Gamma N dt \\
\frac{dN}{N} &= -\Gamma dt \\
\ln N(t) &= -\Gamma t + C \\
N(t) &= e^{C} e^{-\Gamma t} \\
N(t=0) &= e^{C} \\
N(t) &= N(0) e^{-\Gamma t}
\end{aligned}
$$

Decay rate

Because most particles can decay by several different routs, we define the total decay rate as the sum of the individual decay rates.

Formula

$$\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i$$

Branching Ratio

Definition

Branching Ratio $\mathcal{BR}$ = the fraction of all particles of the given type that decay by each mode.

- Branching ratio also called branching fraction.
- For i^{th} decay mode:

$$\mathcal{BR} = \frac{\Gamma_i}{\Gamma_{tot}}$$

Fermi's Golden Rule for Decays

Definition

$$\text{Transition rate} = \frac{2\pi}{\hbar}\left|\text{amplitude}\right|^2 \times (\text{phase space})$$

- Amplitude (matrix element): contains all the *dynamical* information about a process. It can be evaluated from Feynman diagrams using Feynman rules.
- Phase space (density of final states): contains all the *kinetical* information. It depends on the masses, energies, and momenta, etc.

Fermi's Golden Rule for Decays

Suppose particle 1 (at rest) decays into several other particles $2, 3, 4, \cdots, n$.

$$1 \rightarrow 2 + 3 + 4 + \cdots + n$$

, where $p_1 = (m_1 c, \vec{0})$. The total decay rate (transition rate) is given by

Formula

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - \cdots - p_n) \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2 c^2)\theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

The integral sign in total decay rate actually stands for $4(n-1)$ integration.

Fermi's Golden Rule for Decays

$\mathcal{M}$	amplitude
m_j	mass of i^{th} particle
p_j	four moment of i^{th} particle
S	statistical factor $(1/s!)$
$\delta(x)$	constrain: E and $\mathbf{p}$ are conserved. $(E_j^2 = \mathbf{p}_j^2 c^2 + m_j^2 c^4)$
$\theta(x)$	constrain: outgoing energy is positive $(p_j^0 = E_j/c > 0)$.

Example

1. $\Lambda \to p + \pi^-$: $\mathrm{S} = 1$
2. $\pi^0 \to \gamma + \gamma$: $\mathrm{S} = 1/2! = 1/2$
3. $a \to b + b + c + c + c$: $\mathrm{S} = (1/2!)(1/3!) = 1/12$

Fermi's Golden Rule for Decays

Some useful formulas

Formula

- *Delta function*

$$\int f(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{a})d^3\mathbf{x} = f(\mathbf{a})$$

$$\delta(x^2-a^2) = \frac{1}{2a}[\delta(x-a)+\delta(x+a)]$$

- *Heaviside step function*

$$\theta(x) = \begin{cases} 0 & , x < 0 \\ 1 & , x > 0 \end{cases}$$

Fermi's Golden Rule for Decays

Let's consider one particle integral

$$\int 2\pi\delta(p^2 - m^2c^2)\theta(p^0)\frac{d^4p}{(2\pi)^4}$$

$$= \int \delta\{[(p^0)^2 - \mathbf{p}^2] - m^2c^2\}\theta(p^0)\frac{dp^0 d^3\mathbf{p}}{(2\pi)^3}$$

$$= \int \frac{1}{2\sqrt{\mathbf{p}^2 + m^2c^2}}\left\{\delta\left(p^0 - \sqrt{\mathbf{p}^2 + m^2c^2}\right) + \delta\left(p^0 + \sqrt{\mathbf{p}^2 + m^2c^2}\right)\right\}$$

$$\times\, \theta(p^0)\frac{dp^0 d^3\mathbf{p}}{(2\pi)^3}$$

$$= \int \left[\frac{1}{2\sqrt{\mathbf{p}^2 + m^2c^2}}\frac{d^3\mathbf{p}}{(2\pi)^3}\right]\delta\left(p^0 - \sqrt{\mathbf{p}^2 + m^2c^2}\right)dp^0$$

$$= \int \frac{1}{2\sqrt{\mathbf{p}^2 + m^2c^2}}\frac{d^3\mathbf{p}}{(2\pi)^3}, \ \text{with } p^0 = \sqrt{\mathbf{p}^2 + m^2c^2}.$$

Fermi's Golden Rule for Decays

The total decay rate reduces to

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - \cdots - p_n) \times \prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2 c^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

, with $p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2 c^2}$.

Two-particle Decays

For two-particle decays $1 \to 2 + 3$ with particle 1 at rest, the decay rate is

$$
\begin{aligned}
\Gamma &= \frac{S}{2\hbar m_1} \int \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3) \\
&\quad \times \frac{1}{2\sqrt{\mathbf{p}_2^2 + m_2^2 c^2}} \frac{d^3\mathbf{p}_2}{(2\pi)^3} \times \frac{1}{2\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}} \frac{d^3\mathbf{p}_3}{(2\pi)^3} \\
&= \frac{S}{32\pi^2 \hbar m_1} \int \int |\mathcal{M}|^2 \frac{\delta^4(p_1 - p_2 - p_3)}{\sqrt{\mathbf{p}_2^2 + m_2^2 c^2}\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3 \\
&= \frac{S}{32\pi^2 \hbar m_1} \int \int |\mathcal{M}|^2 \frac{\delta(p_1^0 - p_2^0 - p_3^0)\delta^3(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)}{\sqrt{\mathbf{p}_2^2 + m_2^2 c^2}\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3 \\
&= \frac{S}{32\pi^2 \hbar m_1} \int \int |\mathcal{M}|^2 \frac{\delta(m_1 c - p_2^0 - p_3^0)\delta^3(0 - \mathbf{p}_2 - \mathbf{p}_3)}{\sqrt{\mathbf{p}_2^2 + m_2^2 c^2}\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3
\end{aligned}
$$

$$\Gamma = \frac{S}{32\pi^2\hbar m_1}\int\left\{\int |\mathcal{M}|^2 \frac{\delta\left(m_1 c - \sqrt{\mathbf{p}_2^2 + m_2^2c^2} - \sqrt{\mathbf{p}_3^2 + m_3^2c^2}\right)}{\sqrt{\mathbf{p}_2^2 + m_2^2c^2}\sqrt{\mathbf{p}_3^2 + m_3^2c^2}} d^3\mathbf{p}_2\right\}$$

$$\times\ \delta^3(\mathbf{p}_2 + \mathbf{p}_3)d^3\mathbf{p}_3$$

$$= \frac{S}{32\pi^2\hbar m_1}\int |\mathcal{M}|^2 \frac{\delta\left(m_1 c - \sqrt{\mathbf{p}_2^2 + m_2^2c^2} - \sqrt{\mathbf{p}_2^2 + m_3^2c^2}\right)}{\sqrt{\mathbf{p}_2^2 + m_2^2c^2}\sqrt{\mathbf{p}_2^2 + m_3^2c^2}} d^3\mathbf{p}_2$$

where we use $\mathbf{p}_3 = -\mathbf{p}_2$ because of the $\int f(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{a})d^3\mathbf{x} = f(\mathbf{a})$

We use the spherical coordinates in momentum space, $\mathbf{p}_2 \to (p, \theta, \phi)$, $d^3\mathbf{p}_2 \to p^2 \sin\theta \; dp \; d\theta \; d\phi$:

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \int_0^\infty \int_0^\pi \int_0^{2\pi} |\mathcal{M}(p)|^2 \frac{\delta\left(m_1 c - \sqrt{p^2 + m_2^2 c^2} - \sqrt{p^2 + m_3^2 c^2}\right)}{\sqrt{p^2 + m_2^2 c^2}\sqrt{p^2 + m_3^2 c^2}}$$

$$\times\, p^2 \sin\theta \; dp \; d\theta \; d\phi$$

$$= \frac{S}{8\pi \hbar m_1} \int_0^\infty |\mathcal{M}(p)|^2 \frac{\delta\left(m_1 c - \sqrt{p^2 + m_2^2 c^2} - \sqrt{p^2 + m_3^2 c^2}\right)}{\sqrt{p^2 + m_2^2 c^2}\sqrt{p^2 + m_3^2 c^2}} p^2 dp$$

Two-particle Decays

Changing variable, we set

$$u = \sqrt{p^2 + m_2^2 c^2} + \sqrt{p^2 + m_3^2 c^2}$$

$$\begin{aligned}
\frac{du}{dp} &= \frac{d}{dp}(p^2 + m_2^2 c^2)^{1/2} + \frac{d}{dp}(p^2 + m_3^2 c^2)^{1/2} \\
&= \frac{1}{2}(p^2 + m_2^2 c^2)^{-1/2}(2p) + \frac{1}{2}(p^2 + m_3^2 c^2)^{-1/2}(2p) \\
&= p\Big[\frac{1}{\sqrt{p^2 + m_2^2 c^2}} + \frac{1}{\sqrt{p^2 + m_3^2 c^2}}\Big] \\
&= p\Big[\frac{\sqrt{p^2 + m_2^2 c^2} + \sqrt{p^2 + m_3^2 c^2}}{\sqrt{p^2 + m_2^2 c^2}\sqrt{p^2 + m_3^2 c^2}}\Big] \\
&= p\Big[\frac{u}{\sqrt{p^2 + m_2^2 c^2}\sqrt{p^2 + m_3^2 c^2}}\Big]
\end{aligned}$$

Two-particle Decays

After moving u to LHS and dp to RHS, we multiply p on both sides.

$$\frac{p}{u}du = \frac{p^2 dp}{\sqrt{p^2 + m_2^2c^2}\sqrt{p^2 + m_3^2c^2}}$$

Then

$$\begin{aligned}\Gamma &= \frac{S}{8\pi\hbar m_1}\int_{(m_2+m_3)c}^{\infty} |\mathcal{M}(p)|^2 \delta(m_1 c - u)\frac{p}{u}du \\ &= \frac{Sp}{8\pi\hbar m_1^2 c}|\mathcal{M}(p)|^2 \\ &= \frac{S|\mathbf{p}_2|}{8\pi\hbar m_1^2 c}|\mathcal{M}(p)|^2\end{aligned}$$

And

$$m_1 c = u = \sqrt{p^2 + m_2^2c^2} + \sqrt{p^2 + m_3^2c^2}$$

Two-particle Decays

Hence p is

$$p = |\mathbf{p}_2| = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}$$

If $m_1 < m_2 + m_3$, $m_1 c$ is less than the lower bound, $(m_2 + m_3)c$, of integral, then

$$\begin{aligned}\Gamma &= \frac{S}{8\pi\hbar m_1}\int_{(m_2+m_3)c}^{\infty} |\mathcal{M}(p)|^2 \delta(m_1 c - u)\frac{p}{u}du \\ &= 0\end{aligned}$$

This means a particle cannot decay into heavier secondaries.

Two-particle Decays

In two-particle decays, we don't have to know the functional form of $\mathcal{M}$. However, when there are more than two particles in the final state, the integral cannot be done until we know the specific functional form of $\mathcal{M}$.

Formula

- *Golden rule for decays*

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - \cdots - p_n)$$
$$\times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2 c^2)\theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

- *Two-particle decays*

$$\Gamma = \frac{S|\mathbf{p}_2|}{8\pi\hbar m_1^2 c} |\mathcal{M}(p)|^2$$