Sample problems

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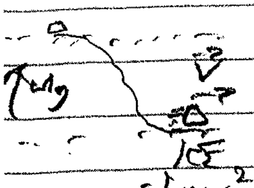
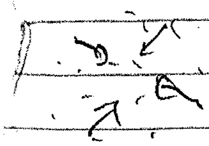
HEP r osc. talk
Skubic F12

11.1

Estimate the lifetime of the sun assuming (as lord Kelvin did) that the source of the energy radiated is gravity.



Stars form from giant H I interstellar medium clouds that are thermally supported against gravity until some sort of compression (say, due to a SN shock wave) triggers collapse. Then, like a ball falling down a hill, the infalling particles give up gravitational P.E. for kinetic energy. It was thought that this KE could support the sun for some time as it gets slowly radiated away.



time it takes to radiate all its energy away

$$t_\odot = \frac{|E_{internal}|}{L_\odot} = \frac{|E_g|}{L_\odot}$$

we assume GPE is the mechanism by which the sun supports itself here.

For a spherical shell of mass dM_r thickness dr, the GPE due to all the enclosed mass (dE_g) is:

① $$dE_g = -\frac{G M_r^{(\text{within } r)}\, dM_r}{r}$$

→

but for the shells:

(2) $dM_r = 4\pi r^2 dr \cdot \rho$

If we assume a uniform ρ with r:

(3a) $M_r = 4\pi\rho \frac{r^3}{3}$ $\Rightarrow$ (3b) $M(r=R)^2 = \frac{16\pi^2\rho^2 R^6}{9}$

Substituting (2) + (3a) → (1):

$$dE_g = -G\frac{\left[4\pi\rho\frac{r^3}{3}\right] \cdot 4\pi r^2 \rho\, dr}{r}$$

integrating:

$$\Rightarrow E_g = \int_0^R \frac{-G \cdot 4^2\pi^2 r^4 \rho^2\, dr}{3}$$

$$= \frac{-16\pi^2 G\rho^2 R^5}{15}$$

(3b) $\Rightarrow E_g = \frac{-3\,GM^2}{5R}$

For the sun, $M = M_\odot \approx 2\times10^{30}$ kg
$R = R_\odot \approx 7\times10^8$ m
$G = 6.67\times10^{-11}\ m^3 kg^{-1} s^{-2}$

$\Rightarrow |E_g| = 2\times10^{41}$ J

$L = L_\odot \approx 4\times10^{26}$ W

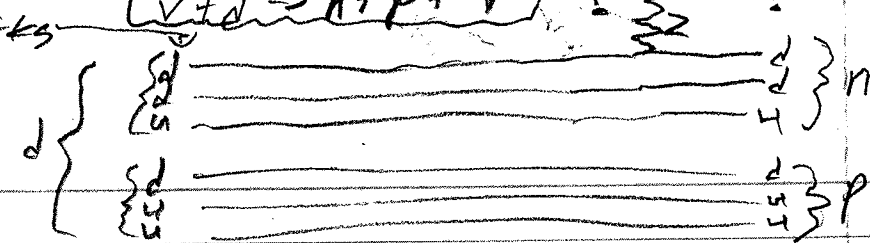
$\Rightarrow \tau_\odot \approx 5\times10^{14}\,s \cdot \frac{1\,m}{60\,s} \cdot \frac{1\,hr}{60\,min} \cdot \frac{1\,d}{24\,hr} \cdot \frac{1\,yr}{365\,d} \approx 2\times10^7$ yrs.

Rocks on Earth have been dated to a few Gyrs ☹
⇒ GPE doesn't power the sun!!

11.3

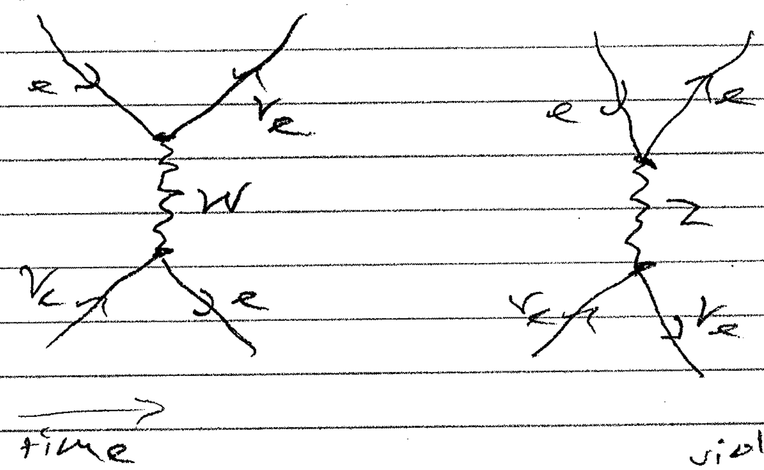
$\nu + d \rightarrow n + p + \nu$ ← Left handed

down quarks

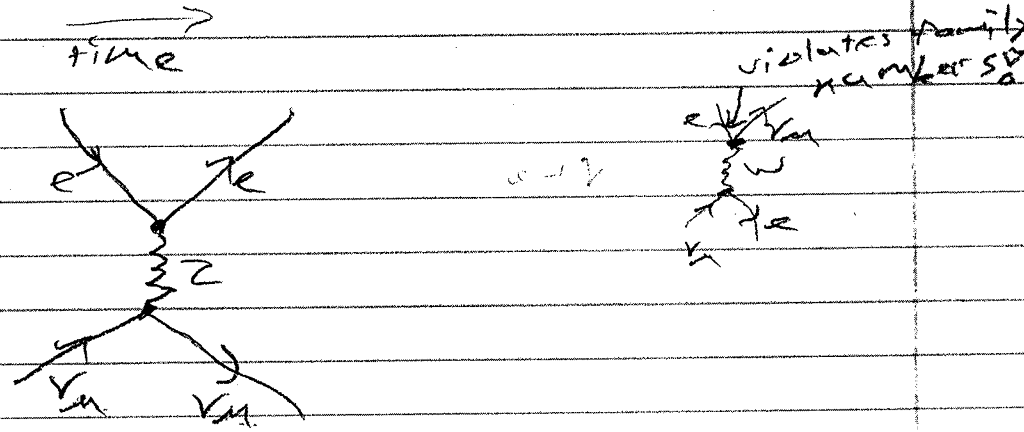


Draw the lowest order diagrams for (a) elastic ν-e^- scattering (b) ν-μ^- scattering and (c) ν-τ^- scattering.

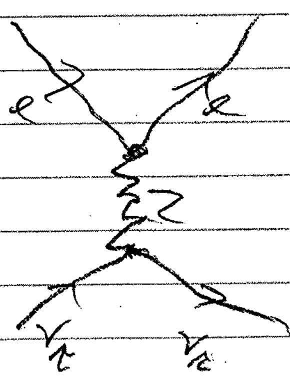
a.



b.



c.



*Note that since weak decays conserve lepton family number (at each vertex), the first ν_e diagram can not happen for the μ or τ. This imbalance makes supernova ν_e bias!

11.5

a. For a relativistic particle:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{mc^2}{E}$$

$$\frac{v^2}{c^2} = 1 - \left(\frac{mc^2}{E}\right)^2$$

$$v = c\left(1 - \frac{m^2c^4}{E^2}\right)^{1/2}$$

Recall $(1+x)^n = 1 + nx + \ldots \approx 1 + nx$ for $x \ll 1$
(binomial expansion)

Here, $E^2 \gg (mc^2)^2$ with $n = 1/2$ & $x = \frac{-m^2c^4}{E^2}$

$$\Rightarrow \quad v \approx c\left(1 - \frac{m^2c^4}{2E^2}\right) \quad (1)$$

also: $\frac{1}{v} \approx \frac{1}{c}\left(1 - \frac{m^2c^4}{2E^2}\right)^{-1}$

Again expanding but with $n = -1$ & $x = \frac{-m^2c^4}{2E^2}$

$$\frac{1}{v} \approx \frac{1}{c}\left(1 + \frac{m^2c^4}{2E^2}\right)$$

b. SN1987A occured in the Large Magellanic Cloud (1.7×10^5 light years from Earth). ν's from electron capture during the collapse of the stars ^{56}Fe core, with energy ranges of 20 MeV

→

to 30 MeV were detected within a 10 s time interval by SuperK. What upper bound on the neutrino mass does this imply, assuming they all started out at the same instant?

In the lab (Earth's) frame, from which we'll do all our work (as the distance to the LMC were measured in Earth's frame):

$$\Delta t = t_2 - t_1 \lesssim 10\,s$$

t_2 ↳ time last ν arrives; t_1 → time first ν arrives

But, for the first ν to leave the LMC (and hence first to arrive at t_1), which travels a distance $L = 1.7 \times 10^5$ Lyrs:

$\Delta x = v \Delta t$

$$t_1 = \frac{L}{v_1}$$

And, for the last ν to leave the LMC (and hence last to arrive at Earth, at t_2):

$$t_2 = \frac{L}{v_2}$$

$$\Rightarrow \Delta t = L\left(\frac{1}{v_2} - \frac{1}{v_1}\right)$$

→

Invoking the result of part a:

$$\Delta t = L\cdot\left\{\left[\frac{1}{c}\left(\cancel{1}+\frac{m_2^2c^4}{2E_2^2}\right)\right]-\left[\frac{1}{c}\left(\cancel{1}+\frac{m_1^2c^4}{2E_1^2}\right)\right]\right\}$$

For $m_1 = m_2 \equiv m$ (for example, both detected ν's are of the same variety - possibly due to detector sensitivities):

$$\Delta t = \frac{L}{2c}\cdot(mc^2)^2\cdot\left(\frac{1}{E_2^2}-\frac{1}{E_1^2}\right)$$

Solving for the ν rest mass:

$$mc^2 = \sqrt{\frac{2c\Delta t}{L}\left(\frac{1}{E_2^2}-\frac{1}{E_1^2}\right)^{-1}}$$

$E_1 > E_2$, as 1 arrives first

We further assume $E_1 = 30\text{ MeV}$ & $E_2 = 2\text{ MeV}$. This will maximize ΔE, giving an upper limit on mc^2, the ν rest mass. This is more evident in the form:

$$mc^2 = \sqrt{\frac{2c\Delta t}{L}\left(\frac{E_1^2-E_2^2}{E_1^2E_2^2}\right)^{-1}}$$

$$mc^2 = E_1E_2\sqrt{\frac{2c\Delta t}{L}\left[\frac{1}{(E_1-E_2)(E_1+E_2)}\right]}$$

$\Rightarrow$ maximizing ΔE minimizes mc_0^2

$\longrightarrow$

so $mc^2 \lesssim E_1 E_2 \sqrt{\frac{2c\Delta t}{L}\left[\frac{1}{(E_1-E_2)(E_1+E_2)}\right]}$

with $L = 1.7\times10^5 \text{Ly} \cdot \frac{9.5\times10^{15}\text{m}}{1 \text{ Ly}} = 1.6\times10^{21}\text{m}$

$\Delta t = 10 s$
$c = 3\times10^8 m/s$
$E_1 = 30 MeV$
$E_2 = 20 MeV$

$\Rightarrow mc^2 \lesssim 5.2\times10^{-5} MeV$

$\boxed{mc^2 \lesssim 52 \text{ eV}}$

$\Rightarrow$ The rest mass of a neutrino is tiny, but nonzero!

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Theory of oscillations [Griffiths 11.2]

Experiments confirm neutrinos oscillate as the transverse space (in time). Here we consider a simplified 2 flavor model (of ν_e & ν_μ)

If neutrinos oscillate, the state of the ν_e & ν_μ cannot be eigenfunctions of the hamiltonian. Steady state solutions (those who's probability does not evolve in time) are solutions to the (non-relativistic) Schrodinger equation. Such solutions permit energy eigenstates, equivalently called "mass" eigenstates. We introduce the orthogonal mass eigenstates (there are two possible in 2D, as is such with 2 flavors):

eqn. (Griffiths 11.1)

$$\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_\mu\\ \nu_e\end{pmatrix}$$

$$|\nu_1\rangle = \cos\theta|\nu_\mu\rangle - \sin\theta|\nu_e\rangle$$
$$|\nu_2\rangle = \sin\theta|\nu_\mu\rangle + \cos\theta|\nu_e\rangle$$

This is analogous to the Cabibbo matrix — where the sine and cosine coefficients have been introduced to represent the possibility of unequal mixing between the states $|\nu_e\rangle$ & $|\nu_\mu\rangle$, with mixing angle θ. That is to say, for some time ~~t~~, there may be preferentially a higher probability for one flavor neutrino over the other. The are

⟶

ways, other than by a rotation, to invoke such a generalization.

Departing breifly from Griffiths, I invert these equations (equivalently, counter rotating): *See scratch page attached

these are not stationary states! ⇒ $P(\nu_\mu)$ will fluctuate in time!

$$\begin{cases} |\nu_e\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle \\ |\nu_\mu\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle \end{cases}$$

If initially, as for the case of the pp chain, we start out with an electron neutrino our neutrino's initial state has the form:

$$|\nu(t=0)\rangle = |\nu_e(t=0)\rangle = -\sin\theta\,|\nu_1(t=0)\rangle + \cos\theta\,|\nu_2(t=0)\rangle$$

Time evolving, we have for the neutrino:

$$|\nu(t)\rangle = -\sin\theta\,|\nu_1(t)\rangle + \cos\theta\,|\nu_2(t)\rangle$$

$$|\nu(t)\rangle = -\sin\theta\, e^{-iE_1t/\hbar}|\nu_1(t=0)\rangle + \cos\theta\, e^{-iE_2t/\hbar}|\nu_2(t=0)\rangle$$

The probability to find the neutrino as a ν_μ is then:

the ν_μ state is implicit in time,

$$P(\nu_\mu) = |\langle\nu_\mu|\nu(t)\rangle|^2$$

N.B. Griffiths QM eqn 3.77: the projection of our ν state into the ν_μ basis, ψ_{ν_μ}; Griffiths lets $\psi_{\nu_\mu}(t) = \langle\nu_\mu|\nu(t)\rangle$,

$$P(\nu_\mu) = \left|(\cos\theta\langle\nu_1| + \sin\theta\langle\nu_2|)\cdot\left(-\sin\theta\, e^{-iE_1t/\hbar}|\nu_1\rangle + \cos\theta\, e^{-iE_2t/\hbar}|\nu_2\rangle\right)\right|^2$$

Exploiting the orthogonality of $|\nu_e\rangle$ & $|\nu_\mu\rangle$, $\langle\nu_e|\nu_\mu\rangle = \langle\nu_\mu|\nu_e\rangle^* = 0$

$$\Rightarrow P(\nu_\mu) = \Big| -\cos\theta \sin\theta\, e^{-iE_1 t/\hbar} \underbrace{\langle\nu_1|\nu_1\rangle}_{1} + \sin\theta\cos\theta\, e^{-iE_2 t/\hbar} \underbrace{\langle\nu_2|\nu_2\rangle}_{1} \Big|^2$$

$$= (\sin\theta\cos\theta)^2 \left| e^{-iE_2 t/\hbar} - e^{-iE_1 t/\hbar} \right|^2$$

$$= (\sin\theta\cos\theta)^2 \left(e^{+iE_2 t/\hbar} - e^{+iE_1 t/\hbar} \right)\left(e^{-iE_2 t/\hbar} - e^{-iE_1 t/\hbar} \right)$$

* note this is again consistent with Griffiths, pg 391.

recall $\sin(2\theta) = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$

$$\Rightarrow \sin\theta\cos\theta = \frac{\sin(2\theta)}{2}$$

$$= \frac{\sin^2(2\theta)}{4} \underbrace{\left(1 - e^{i(E_2 - E_1)t/\hbar} - e^{i(E_1 - E_2)t/\hbar} + 1 \right)}_{\left[2 - \left(e^{i(E_2 - E_1)t/\hbar} + e^{-i(E_2 - E_1)t/\hbar} \right) \right]}$$

$$= \frac{\sin^2(2\theta)}{4} \left(2 - 2\cdot\cos\left[\frac{(E_2 - E_1)t}{\hbar} \right] \right)$$

(half angle formula)

From wikipedia: $\sin^2\frac{\phi}{2} = \frac{1 - \cos\phi}{2} \Rightarrow 2(1 - \cos) = 4\sin^2\frac{\phi}{2}$

$$= \frac{\sin^2(2\theta)}{\cancel{4}} \cdot \cancel{4}\sin^2\left[\frac{(E_2 - E_1)t}{2\hbar} \right]$$

$$\Rightarrow P(\nu_\mu) = P_{\nu_e \to \nu_\mu} = \left[\sin 2\theta \cdot \sin\left(\frac{E_2 - E_1}{2\hbar} t \right) \right]^2 \longrightarrow$$

So, the probability of observing an oscillation evolves in time. Also, the probability depends on the difference in energys of the two mass eigenstates.

For a relativistic particle:

$$E^2 = |\vec{p}|^2c^2 + m^2c^4 = |\vec{p}|^2c^2\left(1 + \frac{m^2c^2}{|\vec{p}|^2}\right)$$

$$\Rightarrow E = |\vec{p}|\,c\left(1 + \frac{m^2c^2}{|\vec{p}|^2}\right)^{1/2} \underset{\text{I}}{\approx} |\vec{p}|c$$

binomially expanding: $(1+x)^n = 1 + nx + \ldots$

here $x = \frac{m^2c^2}{|\vec{p}|^2} \ll 1$ [for the relativistic particle)

$$\Rightarrow E \approx |\vec{p}|c\left(1 + \frac{m^2c^2}{2|\vec{p}|^2}\right)$$

$$E \approx |\vec{p}|c + \frac{m^2c^3}{2|\vec{p}|}$$

For either mass eigenstate, the 3 momentum of the particle is identical.

$$\Rightarrow E_2 - E_1 = \frac{m_2^2c^3 - m_1^2c^3}{2|\vec{p}|}$$

but since (for the Rel. particle) $E \approx pc$, $|\vec{p}| = \frac{E}{c}$: (I)

$$E_2 - E_1 \approx \frac{(m_2^2 - m_1^2)\,c^4}{2E}$$

$\longrightarrow$

So, we can write the probability of an oscillation as:

$$P_{\nu_e \to \nu_\mu} = \left\{ \sin(2\theta) \sin\left[\frac{(m_2^2 - m_1^2)c^4}{2E \cdot 2\hbar} t \right] \right\}^2$$

But we can also write the oscillation in time as an oscillation over space, since the ν travels at $\nu \approx c$, in the lab frame the distance transversed (z) in a time t is:

$$z = ct, \text{ or } t = \frac{z}{c}$$

$$\Rightarrow P_{\nu_e \to \nu_\mu} = \left\{ \sin(2\theta) \sin\left[\frac{(m_2^2 - m_1^2)c^3 \cdot z}{4\hbar E} \right] \right\}^2$$

Final comments:

- oscillations occur over time or through space, not at one point in spacetime as is typical of other processes (strong, weak, EM interactions
- Sensitive to the difference in the square of the masses, or mass eigenstate energys.
- After a distance $z = \frac{2\pi\hbar E}{(m_2^2 - m_1^2)c^3}$ there is a maximum probability of an oscillation.
- If $M_2 = M_1$, no oscillation occurs. $\Rightarrow$ masses cannot be zero!
- the mixing angle (θ) must be nonzero!

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* scratch work to check:

(1) $|V_1\rangle = \cos\theta\,|V_\mu\rangle - \sin\theta\,|V_e\rangle$

(2) $|V_2\rangle = \sin\theta\,|V_\mu\rangle + \cos\theta\,|V_e\rangle$

$$(1) \Rightarrow \frac{-1}{\sin\theta}|V_1\rangle = \frac{-\cos\theta}{\sin\theta}|V_\mu\rangle + |V_e\rangle$$

(3) $$|V_e\rangle = \frac{-1}{\sin\theta}|V_1\rangle + \frac{\cos\theta}{\sin\theta}|V_\mu\rangle$$

(4) $$(2) \Rightarrow |V_\mu\rangle = \frac{1}{\sin\theta}|V_2\rangle - \frac{\cos\theta}{\sin\theta}|V_e\rangle$$

(5) $$(4) \Rightarrow (3): |V_e\rangle = \frac{-1}{\sin\theta}|V_1\rangle + \frac{\cos\theta}{\sin\theta}\left[\frac{1}{\sin\theta}|V_2\rangle - \frac{\cos\theta}{\sin\theta}|V_e\rangle\right]$$

$\sin^2\theta \cdot (5) \Rightarrow (6)$ $$\sin^2\theta\,|V_e\rangle + \cos^2\theta\,|V_e\rangle = -\sin\theta\,|V_1\rangle + \cos\theta\,|V_2\rangle$$

(7a) $$\Rightarrow |V_e\rangle = -\sin\theta\,|V_1\rangle + \cos\theta\,|V_2\rangle$$

$$(7a) \Rightarrow (4): |V_\mu\rangle = \frac{1}{\sin\theta}|V_2\rangle - \frac{\cos\theta}{\sin\theta}\left[-\sin\theta\,|V_1\rangle + \cos\theta\,|V_2\rangle\right]$$

$$= +\cos\theta\,|V_1\rangle + \frac{1}{\sin\theta}\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}|V_2\rangle$$

(7b) $$|V_\mu\rangle = \cos\theta\,|V_1\rangle + \sin\theta\,|V_2\rangle$$