 S. Cairns

Golden Rule for Scattering

Suppose 2 particles collide producing n particles

$$1 + 2 \rightarrow 3 + 4 + \ldots n$$

The cross section is given by:

$$d\sigma = |M|^2 \frac{\hbar^2 S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \left[\left(\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{c\, d^3\vec{p}_4}{(2\pi)^3 2E_4}\right)\cdots\left(\frac{c\, d^3\vec{p}_n}{(2\pi)^3 2E_n}\right)\right]$$
$$\times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 \cdots - p_n)$$

where $p_i = (E_i/c, \vec{p}_i)$ is the four-momentum of the ith particle of mass m_i and $E_i = c\sqrt{m_i^2 c^2 + |\vec{p}_i|^2}$. S is a statistical factor ($1/j!$ for each group of j identical particles in the final state.

Note the delta function enforces conservation of energy and momentum

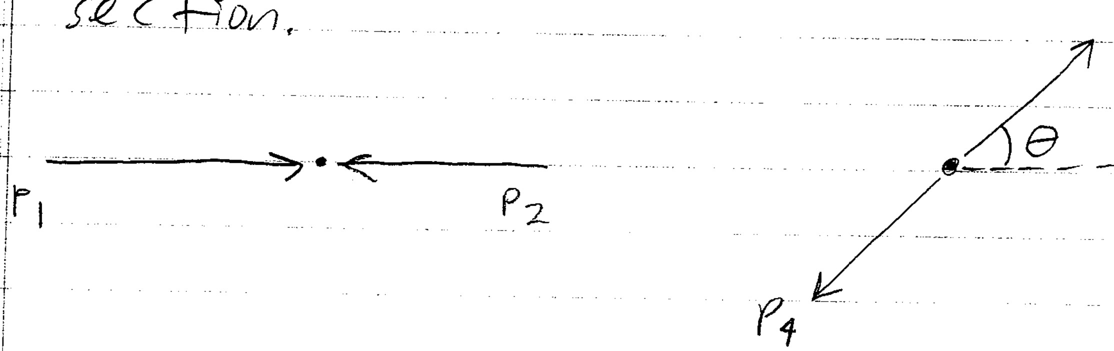
This equation determines the cross section for a process where the three-momentum of particle 3 lies within $d^3\vec{p}_3$, where particle 4 lies between $d^3\vec{p}_4$. In a typical situation we only study the angle particle 3 emerges at. In that case we integrate over all the other momenta and over the magnitude of $\vec{P}_3$. What is left gives us the differential cross section, $d\sigma/d\Omega$, for the scattering of particle 3 into the solid angle $d\Omega$.

Example: Two Body Scattering in the CM Frame

We have,

$$1 + 2 \rightarrow 3 + 4$$

in the CM frame. If the amplitude is M, calculate the differential cross section.



In CM $\vec{P}_2 = -\vec{P}_1$, therefore

$$P_1 \cdot P_2 = \frac{E_1 E_2}{c^2} - \vec{P}_1 \cdot \vec{P}_2 = \frac{E_1 E_2}{c^2} - \vec{P}_1 \cdot (-\vec{P}_1)$$

$$= \frac{E_1 E_2}{c^2} + |\vec{P}_1|^2$$

We can now rewrite one of the terms in $d\sigma$

$$(P_1 \cdot P_2)^2 - (m_1 m_2 c^2)^2 = \left(\frac{E_1 E_2}{c^2} + P_1^2\right)^2 - m_1^2 m_2^2 c^4$$

$$= \frac{E_1^2 E_2^2}{c^4} + \frac{2 E_1 E_2}{c^2} + P_1^4 - m_1^2 \left(E_2^2 - P_1^2 c^2\right)$$

Aside $m_2^2 c^4 = E_2^2 - p_2^2 c^2 = E_2^2 - p_1^2 c^2$

$$m_1^2 = \frac{E_1^2}{c^4} - \frac{p_1^2}{c^2}$$

$$= \frac{E_1^2 E_2^2}{c^4} + \frac{2E_1E_2}{c^2} + p_1^4 - \left(\frac{E_1^2}{c^4} - \frac{p_1^2}{c^2}\right)\left(E_2^2 - p_1^2 c^2\right)$$

$$= \frac{E_1^2 E_2^2}{c^4} + \frac{2E_1E_2}{c^2} + p_1^4 - \frac{E_1^2 E_2^2}{c^4} + \frac{p_1^2}{c^2} E_1^2 + \frac{p_1^2}{c^2} E_2^2 - p_1^4$$

$$= \frac{p_1^2}{c^2}\left(E_1^2 + 2E_1E_2 + E_2^2\right) = \frac{p_1^2}{c^2}(E_1 + E_2)^2$$

Therefore,

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = \sqrt{\frac{\vec{p}_1^{\,2}}{c^2}(E_1 + E_2)^2} = \frac{|\vec{p}_1|}{c}(E_1 + E_2)$$

Now we have,

$$d\sigma = \frac{\hbar^2}{4} \frac{S|M|^2 c}{|\vec{p}_1|(E_1+E_2)} \left(\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{c\, d^3\vec{p}_4}{(2\pi)^3 2E_4}\right)(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$= \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2 c}{|\vec{p}_1|(E_1+E_2)} \frac{d^3\vec{p}_3\, d^3\vec{p}_4}{E_3 E_4} \delta^4(p_1 + p_2 - p_3 - p_4)$$

Rewriting the delta function,

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta\left(\frac{E_1 + E_2 - E_3 - E_4}{c}\right)\delta^3(\vec{p}_1 + \cancel{\vec{p}_2}^{-\vec{p}_1} - \vec{p}_3 - \vec{p}_4)$$

$$= \delta\left(\frac{E_1 + E_2 - E_3 - E_4}{c}\right)\delta^3(-\vec{p}_3 - \vec{p}_4)$$

Now we need to write the outgoing energies in terms of $\vec{p}_3$ and $\vec{p}_4$. Then we will integrate over $\vec{p}_4$

Note: Notation of $d\sigma$ stays even after integration

$$d\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2 c}{|\vec{p}_1|(E_1+E_2)} \frac{d^3\vec{p}_3\, d^3\vec{p}_4}{c^2\sqrt{m_3^2c^2+p_3^2}\sqrt{m_4^2c^2+p_4^2}}\, \delta\left[\frac{(E_1+E_2)}{c} - \sqrt{m_3^2c^2+p_3^2} - \sqrt{m_4^2c^2+p_4^2}\right]$$

$$\times\, \delta^3(-\vec{p}_3 - \vec{p}_4)$$

$$d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|M|^2 c}{|\vec{p}_1|(E_1+E_2)} \frac{\delta\left((E_1+E_2)/c - \sqrt{m_3^2c^2+p_3^2} - \sqrt{m_4^2c^2+p_3^2}\right)}{\sqrt{m_3^2c^2+p_3^2}\sqrt{m_4^2c^2+p_3^2}}\, d^3\vec{p}_3$$

However since $|M|^2$ depends on the direction and magnitude of $\vec{p}_3$ we cannot carry out the angular integration $d\Omega$.

$$d^3\vec{p}_3 = \rho^2\, d\rho\, d\Omega \;;\; \rho = |\vec{p}_3| \quad d\Omega = \sin\theta\, d\theta\, d\phi$$

We now have

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{p}_1|} \int_0^\infty |M|^2 \frac{\delta\left((E_1+E_2)/c - \sqrt{m_3^2c^2+\rho^2} - \sqrt{m_4^2c^2+\rho^2}\right)}{\sqrt{m_3^2c^2+\rho^2}\sqrt{m_4^2c^2+\rho^2}}\, \rho^2\, d\rho$$

Change of variables:

$$E = c\left(\sqrt{m_3^2c^2+\rho^2} + \sqrt{m_4^2c^2+\rho^2}\right)$$

$$dE = c\left(\frac{2\rho}{2\sqrt{m_3^2c^2+\rho^2}} + \frac{2\rho}{2\sqrt{m_4^2c^2+\rho^2}}\right) d\rho$$

$$= c\rho\left(\frac{\sqrt{m_4^2c^2+\rho^2} + \sqrt{m_3^2c^2+\rho^2}}{\sqrt{m_3^2c^2+\rho^2}\sqrt{m_4^2c^2+\rho^2}}\right) d\rho$$

$$= \frac{E\rho}{\sqrt{m_3^2c^2+\rho^2}\sqrt{m_4^2c^2+\rho^2}}\, d\rho$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{p}_i|} \int_{(m_3+m_4)c^2}^{\infty} |M|^2 \frac{p}{E} \underbrace{\delta\left(\frac{(E_1+E_2)}{c} - \frac{E}{c}\right)}_{\substack{\delta\left(\frac{1}{c}\left[(E_1+E_2) - E\right]\right) \\ c\,\delta\left((E_1+E_2) - E\right)}} dE$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S}{(E_1+E_2)|\vec{p}_i|} \int_{(m_3+m_4)c^2}^{\infty} |M|^2 \frac{p}{E} \delta\left((E_1+E_2) - E\right) dE$$

$$= \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2}{(E_1+E_2)^2} \frac{p}{|\vec{p}_i|}$$

Since $p \sim p_3$ we will rewrite $d\sigma/d\Omega$ in a convient form

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2}{(E_1+E_2)} \frac{|\vec{p}_f|}{|\vec{p}_i|}$$

$|\vec{p}_f|$ is the magnitude of either outgoing momentum and $|\vec{p}_i|$ is the magnitude either incoming momentum

Cross sections have dimensions of area - cm^2. A more convenient unit is "barns": 1 barn $= 10^{-24}\ cm^2$. Differential cross sections are given in barns per a steradian. M has units which depend on the number of particles, n. Therefore the dimensions of M are: $M = (mc)^{4-n}$. For example, $A \to B + C$ yields dimensions of momentum. In four body process ($A \to B + C + D$ or $A + B \to C + D$) is dimensionless.