Particle Physics Presentation
4 Nov 09

<u>6.2: The Golden Rule</u>

I. Introduction

a. Erin introduced basic physical quantities necessary to calculate decay rates and scattering cross sections
   i. Amplitude *M* for process
      1. Contains all dynamical information
      2. Calculated by evaluating Feynman diagrams using Feynman rules
   ii. Phase space available
      1. Contains kinematical information
         a. Depends on masses, energies and momenta
         b. Reflects the fact that a given process is more likely to occur given larger phase space factor than smaller phase space factor

b. Fermi's Golden Rule is a way to calculate the transition rate from a discrete energy eigenstate into a continuum of energy eigenstates

$$\text{Transition rate} = \frac{2\pi}{\hbar}|M|^2 \times \text{(phase space)}$$

II. Golden Rules for Decay

a. Consider particle 1 decays into several particles 2, 3, 4, …, n

$$1 \rightarrow 2 + 3 + 4 + \ldots + n \qquad \text{(I)}$$

b. Differential decay rate is given by

$$d\Gamma = |M|^2 \frac{S}{2\hbar m_1}\left[\left(\frac{cd^3\vec{p}_2}{(2\pi)^3 2E_2}\right)\left(\frac{cd^3\vec{p}_3}{(2\pi)^3 2E_3}\right)\cdots\left(\frac{cd^3\vec{p}_n}{(2\pi)^3 2E_n}\right)\right] \qquad \text{(II)}$$
$$\times(2\pi)^4\delta^4(p_1 - p_2 - \ldots - p_n)$$

   i. $p_i = (\frac{E_i}{c}, \vec{p}_i)$ = 4-momentum and contains mass so $E_i^2 - \vec{p}_i^2 c^2 = m_i^2 c^4$
   ii. Delta function enforces conservation of energy and momentum
   iii. Decaying particle presumed to be at rest, so $p_1(m_1 c, \vec{0})$
   iv. *S* is a product of statistical factors: $1/j!$ for each group of *j* identical particles in the final state

v. Determines decay in which the 2-momentum of particle 2 lies in the range $d^3\vec{p}_2$ about the value $\vec{p}_2$ and so on

c. Integrate over all outgoing momenta to get total decay rate Γ for mode in question (Eqn. I)
   i. Consider the case where there are only 2 particles in the final state
   ii. The total decay rate is

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{c}{4\pi}\right)^2 \frac{1}{2}\int \frac{|M|^2}{E_2 E_3}\delta^4(p_1 - p_2 - p_3)d^3\vec{p}_2 d^3\vec{p}_3$$

      1. Amplitude *M* is a function of $\vec{p}_2$ and $\vec{p}_3$
      2. Can solve integral explicitly without knowing functional form of *M*

d. Example 6.5:
   i. Special case where decay products are massless
   ii. Particle of mass *m* decays into two massless particles (say, $\pi^0 \to \gamma + \gamma$)
   iii. Let the amplitude for the process be $M(\vec{p}_2, \vec{p}_3)$
   iv. Rewrite delta function using the fact that $E_1 = mc^2$ and $\vec{p}_1 = \vec{0}$

$$p_1 = \left(\frac{E_1}{c}, \vec{0}\right) = \frac{mc^2}{c} = mc$$

$$p_2 = \left(\frac{E_2}{c}, \vec{p}_2\right)$$

$$p_3 = \left(\frac{E_3}{c}, \vec{p}_3\right)$$

$$\delta^4(p_1 - p_2 - p_3) = \delta(mc - \frac{E_2}{c} - \frac{E_3}{c})\delta^3(-\vec{p}_2 - \vec{p}_3)$$

   v. But $m_2 = m_3 = 0 \Rightarrow E_2 = |\vec{p}_2|c, E_3 = |\vec{p}_3|c$, so we have

$$\delta(mc - \frac{E_2}{c} - \frac{E_3}{c}) = \delta(mc - |\vec{p}_2| - |\vec{p}_3|)$$

   vi. The decay rate is

$$\Gamma = \frac{S}{\hbar m}\left(\frac{c}{4\pi}\right)^2 \frac{1}{2}\frac{1}{c^2}\int \frac{|M|^2}{|\vec{p}_2||\vec{p}_3|} \times \delta(mc - |\vec{p}_2| - |\vec{p}_3|)\delta^3(-\vec{p}_2 - \vec{p}_3)d^3(\vec{p}_2)d^3(\vec{p}_3)$$

$$= \frac{S}{\hbar m}\left(\frac{1}{4\pi}\right)^2 \frac{1}{2}\int \frac{|M|^2}{|\vec{p}_2||\vec{p}_3|} \times \delta(mc - |\vec{p}_2| - |\vec{p}_3|)\delta^3(-\vec{p}_2 - \vec{p}_3)d^3(\vec{p}_2)d^3(\vec{p}_3)$$

   vii. Now we can use the $\delta^3(-\vec{p}_2 - \vec{p}_3)$ to replace $\vec{p}_3$ with $-\vec{p}_2$ (which reflects conservation of momentum)

$$\Gamma = \frac{S}{2(4\pi)^2 \hbar m}\int \frac{|M|^2}{|\vec{p}_2|^2}\delta(mc - 2|\vec{p}_2|)d^3(\vec{p}_2)$$

viii. Recall that $|M|^2$ is a function of $|\vec{p}_2|$ (since it is a scalar)

ix. Switching to spherical coordinates

$d^3\vec{p}_2 = |\vec{p}_2|^2 d|\vec{p}_2| d\Omega; \int d\Omega = \int \sin\theta d\theta d\phi = 4\pi$

$$\Gamma = \frac{S}{32\pi^2\hbar m} 4\pi \int_0^\infty \frac{|M|^2}{|\vec{p}_2|^2} |\vec{p}_2|^2 \delta(mc - 2|\vec{p}_2|) d|\vec{p}_2|$$

$$= \frac{S}{8\pi\hbar m} \int_0^\infty |M|^2 \delta(mc - 2|\vec{p}_2|) d|\vec{p}_2|$$

x. Using the identity $\delta(kx) = \frac{1}{|k|}\delta(x)$, we find that

$$\delta(mc - 2|\vec{p}_2|) = \frac{1}{|-2|}\delta\left(\frac{-mc}{2} + |\vec{p}_2|\right) = \frac{1}{2}\delta\left(|\vec{p}_2| - \frac{mc}{2}\right)$$

where

$x = mc - |\vec{p}_2|$

$k = -2$

xi. Then

$$\Gamma = \frac{S}{8\pi\hbar m} \int_0^\infty |M|^2 \frac{1}{2} \delta(|\vec{p}_2| - \frac{mc}{2}) d|\vec{p}_2|$$

$$= \frac{S}{16\pi\hbar m} \int_0^\infty |M|^2 \delta(|\vec{p}_2| - \frac{mc}{2}) d|\vec{p}_2|$$

$$= \frac{S}{16\pi\hbar m} |M|^2$$

with

$\vec{p}_3 = -\vec{p}_2$

$|\vec{p}_2| = \frac{mc}{2}$

e. Example 6.6

i. General case of Example 6.5, where outgoing particles have mass

ii. Amplitude is given

iii. Again, rewrite delta function using the fact that $E_1 = mc^2$ and $\vec{p}_1 = \vec{0}$

$$p_1 = \left(\frac{E_1}{c}, \vec{0}\right) = \frac{mc^2}{c} = mc$$

$$p_2 = \left(\frac{E_2}{c}, \vec{p}_2\right)$$

$$p_3 = \left(\frac{E_3}{c}, \vec{p}_3\right)$$

$$\delta^4(p_1 - p_2 - p_3) = \delta(mc - \frac{E_2}{c} - \frac{E_3}{c})\delta^3(-\vec{p}_2 - \vec{p}_3)$$

iv. But now, $m_2, m_3 \neq 0$

$$E^2 = p^2c^2 + m^2c^4$$

$$\frac{E_2}{c} = \sqrt{p_2^2 + m_2^2c^2}$$

$$\frac{E_3}{c} = \sqrt{p_3^2 + m_3^2c^2}$$

$$\delta(mc - \frac{E_2}{c} - \frac{E_3}{c}) = \delta(mc - \sqrt{p_2^2 + m_2^2c^2} - \sqrt{p_3^2 + m_3^2c^2})$$

v. The decay rate is then

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{c}{4\pi}\right)^2 \frac{1}{2}\int \frac{|M|^2 \delta(m_1c - \sqrt{p_2^2 + m_2^2c^2} - \sqrt{p_3^2 + m_3^2c^2})}{\sqrt{p_2^2 + m_2^2c^2\sqrt{p_3^2 + m_3^2c^2}}}\delta^3(-\vec{p}_2 - \vec{p}_3)d^3(\vec{p}_2)d^3(\vec{p}_3)$$

vi. Solving the $\vec{p}_3$ integral as before by replacing $\vec{p}_3$ with $-\vec{p}_2$

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{c}{4\pi}\right)^2 \frac{1}{2}\int \frac{|M|^2 \delta(m_1c - \sqrt{p_2^2 + m_2^2c^2} - \sqrt{p_2^2 + m_3^2c^2})}{\sqrt{p_2^2 + m_2^2c^2\sqrt{p_2^2 + m_3^2c^2}}}d^3(\vec{p}_2)$$

vii. $|M|^2$ is a function of $|\vec{p}_2|$ and introducing spherical coordinates gives us

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{c}{4\pi}\right)^2 \frac{4\pi}{2}\int_0^\infty \frac{|M|^2 \delta(m_1c - \sqrt{p_2^2 + m_2^2c^2} - \sqrt{p_2^2 + m_3^2c^2})}{\sqrt{p_2^2 + m_2^2c^2\sqrt{p_2^2 + m_3^2c^2}}}|\vec{p}_2|d(\vec{p}_2)$$

$$= \frac{Sc^2}{8\pi\hbar m_1}\int_0^\infty \frac{|M|^2 \delta(m_1c - \sqrt{p_2^2 + m_2^2c^2} - \sqrt{p_2^2 + m_3^2c^2})}{\sqrt{p_2^2 + m_2^2c^2\sqrt{p_2^2 + m_3^2c^2}}}|\vec{p}_2|d(\vec{p}_2)$$

$$= \frac{Sc^2}{8\pi\hbar m_1}\int_0^\infty \frac{|M|^2 \delta(m_1c - \sqrt{\rho^2 + m_2^2c^2} - \sqrt{\rho^2 + m_3^2c^2})}{\sqrt{\rho^2 + m_2^2c^2\sqrt{\rho^2 + m_3^2c^2}}}\rho^2 d\rho$$

$$\rho = |\vec{p}_2|$$

viii. Making a change of variable and define E (representing the total energy of the outgoing particles) to be

$$E \equiv c\left(\sqrt{\rho^2 + m_2^2c^2} + \sqrt{\rho^2 + m_3^2c^2}\right)$$

We can find *dE* by

$$dE = c(\rho^2 + m_2^2c^2)^{1/2} + c(\rho^2 + m_3^2c^2)^{1/2}d\rho$$

$$= \frac{c}{2}(\rho^2 + m_2^2c^2)^{-1/2}\cdot 2\rho + \frac{c}{2}(\rho^2 + m_3^2c^2)^{-1/2}\cdot 2\rho d\rho$$

$$= c\rho\left[(\rho^2 + m_2^2c^2)^{-1/2} + (\rho^2 + m_3^2c^2)^{-1/2}\right]d\rho$$

$$= c\rho\left[\frac{(\rho^2 + m_2^2c^2)^{1/2} + (\rho^2 + m_3^2c^2)^{1/2}}{(\rho^2 + m_2^2c^2)^{1/2}(\rho^2 + m_3^2c^2)^{1/2}}\right]d\rho$$

$$= \frac{E\rho}{\sqrt{\rho^2 + m_2^2c^2}\sqrt{\rho^2 + m_3^2c^2}}d\rho$$

$$\Rightarrow d\rho = \frac{\sqrt{\rho^2 + m_2^2c^2}\sqrt{\rho^2 + m_3^2c^2}}{E\rho}dE$$

ix. Substituting this into the decay function

$$\Gamma = \frac{Sc^2}{8\pi\hbar m_1}\int_0^\infty \frac{|M|^2\delta(m_1c - \sqrt{\rho^2 + m_2^2c^2} - \sqrt{\rho^2 + m_3^2c^2})}{\sqrt{\rho^2 + m_2^2c^2\sqrt{\rho^2 + m_3^2c^2}}}\rho^2 d\rho$$

$$\delta(m_1c - \sqrt{\rho^2 + m_2^2c^2} - \sqrt{\rho^2 + m_3^2c^2}) = \delta(m_1c - \frac{E}{c})$$

$$\Gamma = \frac{Sc^2}{8\pi\hbar m_1}\int_{(m_2+m_3)c^2}^\infty \frac{|M|^2\delta(m_1c - \frac{E}{c})}{\sqrt{\rho^2 + m_2^2c^2\sqrt{\rho^2 + m_3^2c^2}}}\rho^2\frac{\sqrt{\rho^2 + m_2^2c^2}\sqrt{\rho^2 + m_3^2c^2}}{E\rho}dE$$

$$= \frac{Sc^2}{8\pi\hbar m_1}\int_{(m_2+m_3)c^2}^\infty |M|^2\frac{\rho}{E}\delta(m_1c - \frac{E}{c})dE$$

x. Using the identity $\delta(kx) = \frac{1}{|k|}\delta(x)$, we find that

$$\delta(m_1c - \frac{E}{c}) = c\delta(E - m_1c^2)$$

where

$$x = m_1c - \frac{E}{c}$$

$$k = -\frac{1}{c}$$

xi. Then

$$\Gamma = \frac{S}{8\pi\hbar m_1}\int_{(m_2+m_3)c^2}^\infty |M|^2\frac{\rho c}{E}\delta(E - m_1c^2)dE$$

$$= \frac{S\rho_0|M|^2}{8\pi\hbar m_1^2c}; m_1 > m_2 + m_3$$

xii. If $m_1 < m_2 + m_3$, then $\Gamma = 0$. This records the fact that a particle cannot decay into heavier secondaries. Also, ρ_0 is the value of ρ for which $E = m_1 c^2$ and

$$\rho_0 = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$

xiii. This tells us that ρ_0 is the particular value of $|\vec{p}_2|$ that is consistent with conservation of energy

xiv. Ultimately, we find that

$$\Gamma = \frac{S|\vec{p}||M|^2}{8\pi\hbar m_1^2 c}$$

where $|\vec{p}|$ is the magnitude of either outgoing momentum and the amplitude is evaluated at the momenta required by conservation laws.

xv. Note that if $m_2 = m_3 = 0 \Rightarrow |\vec{p}| = \frac{m_1 c}{2}$ and we recover the result from Example 6.5